Quantum Mechanics Qualify Exam, **Part I**, **50 points** **2020 – 10 – 15**

1. [20 points] Consider a particle whose wave function is $\psi(x) = A\sin(p_0 x/\hbar)$.

   (a). Is this wave function an eigenstate of momentum? (5 points)

   (b). Find the expectation value $\langle p \rangle$ of the momentum and the momentum probability distribution. (10 points)

   (c). Calculate the uncertainty Δp of the momentum. (5 points)

2. [15 points] A particle in an infinite square well potential has an initial state vector $|\psi(t=0)\rangle = (|\phi_1\rangle - 2i|\phi_2\rangle)/\sqrt{5}$ where the $|\phi_n\rangle$ are the eigenfunctions of the Hamiltonian operator.

   (a). Find the time evolution of the state vector. (7 points)

   (b). Find the expectation value of the position as a function of time. (8 points)

3. [15 points] Consider a quantum system with a set of energy eigenstates $|E_n\rangle$ where the energies are given by $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$. The system is in the state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\alpha^2/2}}{\sqrt{n!}} |E_n\rangle,$$

   where α is a positive real number.

   (a). Find the probabilities for measuring the energy eigenvalues. (5 points)

   (b). Find the expectation value of the energy. (5 points)

   (c). Find the uncertainty of the energy. (5 points)

Quantum Mechanics Qualifying Exam Part II (50 points) 2020/10/15

1. Consider a particle in a state described by the wave function $\psi = A(e^{i\phi}\sin\theta + \cos\theta)g(r)$, where $\int_0^\infty |g(r)|^2 r^2 dr = 1$ and A is a real constant.
   (a) Calculate the normalization constant A. (6 points).
   (b) Write the wave function in terms of spherical harmonics. (2 points)
   (c) Write down the possible values of a measurement of the z-component L_z of the angular momentum in this state, and calculate the probability of obtaining each of these values. (8 points)
   (d) Calculate the expectation value of L_z. (4 points)
   [Hint: $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$.]

2. Consider a particle with spin quantum number $s = 1$ and ignore all spatial degrees of freedom. The particle is subject to a uniform external magnetic field $\vec{B} = B\hat{x}$, and the Hamiltonian of the system is $H = g\vec{B}\cdot\vec{S}$, where g is a constant.
   (a) In the basis of the $\vec{S}^2$, S_z eigenstates $|s,m>$, write down the raising and lowering operators $S_\pm$ as 3×3 matrices using the relation $S_\pm |s,m> = \hbar\sqrt{s(s+1)-m(m\pm1)}\,|s,m\pm1>$. (6 points)
   (b) Using the definition $S_\pm = S_x \pm iS_y$, write down S_x, S_y, S_z as 3×3 matrices. (9 points)
   (c) Calculate the eigenvalues and corresponding eigenvectors of S_x. (9 points)
   (d) If the particle is initially (at $t=0$) in the state $|1,1>$, find the evolved state of the particle at time $t>0$ in terms of the eigenvectors of S_x. (3 points)
   (e) What is the probability of finding the particle in the state $|1,-1>$ at time $t>0$. (3 points)

Part III

1. [30%] With the Born approximation, calculate (a) the scattering amplitude and (b) the total cross-section for scattering from a Yukawa potential $V(r) = \beta \frac{e^{-\mu r}}{r}$.

2. [35%] Consider a system with a simple 2x2 matrix Hamiltonian $H = H_0 + H_1$, for which

$$H_0 = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \text{ and } H_1 = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}$$

   First find the exact energies of the system $(H_0+H_1)|\Psi\rangle = E\,|\Psi\rangle$, then use perturbation theory to find the energies for the nondegenerate and degenerate cases: $\epsilon_1 \neq \epsilon_2$ and $\epsilon_1 = \epsilon_2$.

3. [35%] (a) Write down the complete singlet and triplet wave functions for 2-electron system occupying 2 space states $\psi_a(\vec{r})$ and $\psi_b(\vec{r})$, and spin states *u* or *d* (write particle 1 before particle 2 to avoid confusion). (b) Write down the energy shift due to the Coulomb interaction $\frac{e^2}{4\pi\epsilon_0|\vec{r}_1 - \vec{r}_2|}$ in the form $\Delta E = J \pm K$ (you need write out the details of *J* and *K*) and argue on physical ground that the energy is lower when the electron spins are parallel (This is the origin of ferromagnetism).

Quantum Mechanics Qualify 2020/03/26
Exam Part I (50 points)

1. (a) [9 points] Let E_0 be the exact ground state energy of the Hamiltonian H. Show that, for any trial state $|\Psi>$

$$\frac{<\Psi|H|\Psi>}{<\Psi|\Psi>} \geq E_0$$

(b) [15 points] Using the trial wavefunction $\Psi = e^{-\alpha x2/2}$ to estimate the ground state energy of a one-dimensional problem with $H = P^2/(2m) + V$ where $V(x) = -aV_0\delta(x)$.

2. A particle of mass m locates between a left and a right infinite wall at x = 0 and x = L, respectively. The particle is in its quantum ground state.

(a) [5 Points] What is the energy of this state?

(b) [7 Points] Assume the right wall extends slowly (adiabatically) from x = L to x = $2L$. How does the expectation value of the particle energy change?

(c) [14 Points] Now assume that this wall suddenly extends from L to $2L$ at speed $v \gg \sqrt{E/m}$;

(1) What happens to the expectation value of energy?

(2) Compute the possibility of particle in the ground state in this system.

Quantum Mechanics Qualify Exam, **Part II,** **50 points** **2020 – 03 – 26**

1. [15 points] Suppose an electron is in a state described by the wave function

$$\Psi = \frac{1}{\sqrt{4\pi}}(e^{i\phi}\sin\theta + \cos\theta)g(r),$$

where

$$\int_0^\infty |g(r)|^2 r^2 dr = 1,$$

and ϕ, θ are the azimuthal and polar angles respectively.

(a) What are the possible results of a measurement of the z-component L_z of the angular momentum of the electron in this state? (5 points)

(b) What is the probability of obtaining each of the possible results in part (a) ? (5 points)

(c) What is the expectation value of L_z ? (5 points)

2. [20 points] At time $t = 0$, the state of the spin-1/2 electron is

$$|\psi(t=0)> = |+>_n$$

with the direction $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The system is allowed to evolve in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. What is the probability that the particle will be measured to have spin up in the y-direction after a time t ?

3. [5 points] Give a brief discussion on the selection rules for changes of j, l, and m for electromagnetic electric dipole transitions.

4. [10 points] Classify the independent states of a system which consists of three weakly interacting subsystems whose angular momenta are $l_1 = l_2 = 1$ and $l_3 = l$, by the value of the total angular momentum L.

Qualifying Exam: Quantum Mechanics
Part 3: 50%

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1. [25%] The nucleus of a deuterium consists of two nucleons: a proton and a neutron. Let the potential energy between the two nucleons have the radial form:

$$V(r) = -Ae^{-r/a} \ , (A = 32\mathrm{MeV}, a = 2.2 \times 10^{-15}\mathrm{m}).$$

Suppose we approximate the ground-state energy of the system by using the wave function:

$$\psi(r) = Ne^{-\lambda r/2a} \ ,$$

where λ is a variational parameter.

   (a) [5%] Determine N in the $\psi(r)$?

   (b) [20%] The reduced mass of the system is $\mu = \frac{m_p m_n}{m_p + m_n} = 469.45\mathrm{MeV/c}^2$. Find the ground-state energy $E(\lambda)$ as a function λ and hence obtain the approximate ground-state energy E_0. (Express E_0 in an algebraic form)

2. [10%] Consider an elastic scattering process of a free particle with a fixed target. Suppose the potential energy $V(r)$ of the particle and the target is of radial form and of finite range. If the incident particle has momentum $\mathbf{p} = \hbar\mathbf{k}$ in the z-direction and the scattered particle has momentum $\hbar\mathbf{k}'$, use the Schrödinger equation to deduce the general form of the full wave function for very large separation r.

3. [15%] Look at the one-dimensional harmonic oscillator and express its Hamiltonian as follows ($\hbar = 1$):

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \frac{\omega}{2}(aa^\dagger + a^\dagger a) \ ,$$

with the creation and annihilation operators, $a^\dagger$ and a, being related to x and p:

$$x = \sqrt{\frac{1}{2m\omega}}(a + a^\dagger) \ , p = \frac{1}{i}\sqrt{\frac{m\omega}{2}}(a - a^\dagger) \ .$$

Suppose we add a perturbation

$$H' = \frac{1}{2}\lambda x^2.$$

Calculate the first-order perturbation of the n-th level $|\psi_n^0\rangle$, which is the eigen-function of H_0: $H_0|\psi_n^0\rangle = (n + \frac{1}{2})\omega|\psi_n^0\rangle$.

Useful Formulae

Laplacian:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \ .$$

The differential equation:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2\right]\chi(\mathbf{r}) = 0$$

has solutions:

$$\chi(\mathbf{r}) = \frac{f(\theta,\phi)}{(2\pi)^3}\frac{e^{\pm ikr}}{r} \ .$$

Quantum Mechanics Qualify Exam, **Part I**, **50 points** **2019 – 10 – 17**

1. [10 points] Use the de Broglie relation between the wavelength and the linear momentum of a non-relativistic particle, and the Planck relation between frequency and the quantum energy, to obtain the phase and group velocities of the wave motion assiciated with a particle with velocity v which is much smaller than the speed of light c.

2. [15 points] Consider an infinite square potential well

$$V(x) = \begin{cases} 0, & -a < x < a \\ \infty, & |x| \geq a. \end{cases}$$

The wave function of a particle trapped in this potential well is found to be

$$\Psi(x) = \begin{cases} A(\cos\frac{\pi x}{2a} + \sin\frac{3\pi x}{a} + \frac{1}{4}\cos\frac{3\pi x}{2a}), & -a < x < a \\ 0, & |x| \geq a. \end{cases}$$

(a) Calculate the coefficient A. (5 points)

(b) If a measurement of the total energy is made, what are the possible results of such a measurement, and what is the probability to measure each of them? (10 points)

3. [13 points] The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where $a, b,$ and c are real numbers (assume $a - c \neq \pm b$). If the system starts out in the state

$$f(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what is $f(t)$?

4. [12 points] Consider a one-dimensional harmonic oscillator of mass m and angular frequency ω that is on its n-th energy level.

(a) Find $\langle x^2 \rangle$. (6 points)

(b) Find the average kinetic energy of a one-dimensional harmonic oscillator if whose total energy is $\frac{7}{2}\hbar\omega$. (6 points)

Quantum Mechanics Qualifying Exam Part II (50 points) 2019/10/17

1. Consider an electron.
   (a) Write down the spin operator $\vec{S}$ in terms of three Pauli matrices. (5 points)
   (b) For an arbitrary direction $\hat{r} = \sin\theta\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right) + \cos\theta\hat{z}$, solve the eigenvalues and eigenvectors of the operator $\hat{r}\cdot\vec{S}$ that represents the spin projection along $\hat{r}$. (10 points)
   (c) If the electron is described by a Hamiltonian that does not depend on spin, and the electron's spin wave function is an eigenstate of S_z with eigenvalue $\hbar/2$. What is the probability of finding the electron in each $\hat{r}\cdot\vec{S}$ eigenstate. (5 points)

2. Consider a spinless particle represented by the wave function $\psi = A(x+y+2z)e^{-\alpha r}$, where $r = \sqrt{x^2+y^2+z^2}$, while A and α are real constants.
   (a) Write the wave function in spherical coordinates. (5 points).
   (b) Write the angular part of the wave function $\psi(\theta,\phi)$ in terms of spherical harmonics and solve for its normalization constant. (10 points)
   (c) Calculate the expectation value of the square of the total angular momentum, L^2. (5 points)
   (d) Calculate the expectation value of the z-component of angular momentum, L_z. (5 points)
   (e) If the z-component of angular momentum were measured, calculate the probability that the result would be $L_z = +\hbar$. (5 points)

   [Hint: $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$, $Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$, $Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$, $Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}$.]

Ph.D. Qualifying Exam: Quantum Mechanics
October 17, 2019
Part 3: 50%

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1. [20%] Consider the following Hamiltonian in **suitable units**:

$$\mathcal{H} = a^\dagger a + \lambda(a + a^\dagger) \ ,$$

where $a^\dagger$ and a are creation and annihilation operators satisfying the commutation relations: $[a, a^\dagger] = 1$, $[a, a] = 0$, $[a^\dagger, a^\dagger] = 0$, and λ is a real constant. Treat $\mathcal{H}^{(1)} = \lambda(a + a^\dagger)$ as a perturbation of the harmonic oscillator $\mathcal{H}^{(0)} = a^\dagger a$.

   (a) [15%] Calculate the ground state energy of $\mathcal{H}$ to second order in perturbation theory in λ.

   (b) [5%] Find the exact energies of $\mathcal{H}$. (You may define new operators $b^\dagger$ and b, which are related to $a^\dagger$ and a by a constant shift.)

2. [25%] Consider the Hamiltonian of the **three-dimensional** simple harmonic oscillator

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m\omega^2 \mathbf{r}^2 \ ,$$

which has ground-state energy $E_0 = \frac{3}{2}\omega$ in units $\hbar = 1$. Suppose you did not know the exact ground-state energy and guessed the normalized trial wave function:

$$\psi(r, a) = \frac{1}{\sqrt{\pi} a^{3/2}} e^{-r/a} \ .$$

What would be the ground-state energy?

3. [5%] Consider the scattering amplitude as a partial wave expansion

$$f(\theta, \varphi) = \frac{1}{k} \sum_\ell (2\ell + 1) e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta) \ ,$$

where δ_ℓ is the phase shift and $P_\ell(\cos\theta)$ is the Legendre polynomial. What is the total cross section σ_{tot}?

Useful Formulae

$$\nabla^2 f(\mathbf{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \ .$$

$$I(\alpha) = \int_0^\infty e^{-\alpha r} dr = \frac{1}{\alpha} \ , \quad \int_0^\infty e^{-\alpha r} r^n dr = \left(-\frac{d}{d\alpha}\right)^n \left(\frac{1}{\alpha}\right) \ .$$

$$\int_0^\pi P_{\ell'}(\cos\theta) P_\ell(\cos\theta) \sin\theta d\theta = \frac{2}{2\ell + 1}\delta_{\ell'\ell} \ .$$

National Cheng Kung University

Physics Ph.D. Qualifying Examination

Quantum Mechanics

Part I (50%)

Q 1. [Total 29%] Let $\Psi(x, t = 0) = A\,\mathrm{e}^{-x^2/a^2}$ be a solution to the Schrödinger equation (for a *freely moving* particle of mass m) at the instant of time $t = 0$. For this instant of time,

(a) [6%] determine the normalization constant A and explain the physical meaning of the normalization requirement [*you may use the fact that* $\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y = \sqrt{\pi}$];

(b) [2%] determine the expectation value of the position x of the particle;

(c) [4%] determine the expectation value of x^2 of the particle;

(d) [3%] determine the expectation value of the momentum p of the particle;

(e) [5%] determine the expectation value of p^2 of the particle;

(f) [2%] what does your answer in part (e) tell you about the expectation value of its energy (Hamiltonian)?

(g) [7%] verify using your answer to part (b)-part (e) that in this case, the Heisenberg uncertainty relation is saturated (i.e., the inequality becomes an equality).

Q 2. [Total 21%] The Schrödinger equation is a linear partial differential equation that is first order in t. Given its solution at any particular instant of time $t = t_0$, the evolution of this solution (wavefunction) as a function of time can be determined exactly. For example, starting from the wavefunction $\Psi(x, t = 0)$ given in **Q 1.**, one can determine its wavefunction for all subsequent values of t. To this end,

(a) [3%] determine the momentum representation $\Phi(0, t)$ of the wavefunction $\Psi(x, t = 0)$ given in **Q 1.**;

(b) [5%] solve the Schrödinger equation to determine the wavefunction of the system in the momentum representation $\Phi(p, t)$ for arbitrary time $t > 0$
[*Hint: your calculation simplifies if you make use of the Schrödinger equation expressed in the momentum representation*];

(c) [3%] using your answer to part (b) or otherwise, determine the wavefunction of the system in the position representation $\Psi(x, t)$ for arbitrary time $t > 0$.

Based on the determined $\Psi(x, t)$,

(d) [3%] evaluate the probability of finding the particle at position(s) $x \leq 0$;

(e) [2%] or otherwise, explain—without evaluating any integral—how you think the expectation value of the Hamiltonian of the particle may, or may not change as a function of time;

(f) [2%] evaluate the expectation value of the position x of the particle;

(g) [3%] explain how you think the uncertainty of the position of this particle may, or may not change as a function of time.

Quantum Mechanics Qualify Exam Part Ⅱ (50 points) 2019/03/28

1. Two quantum particles of mass *m* are attached to the ends of a massless rigid rod of length *a*. The system is free to rotate in three dimensions about the center (but the center point itself is fixed).

   (a) Show that the allowed energies of this rigid rotor are (5 points)

   $$E_n = \frac{\hbar^2 n(n+1)}{ma^2}, \quad for \quad n = 0,\ 1,\ 2,\ \ldots$$

   (b) What are the normalized eigenfunctions for this system? (3 points) What is the degeneracy of the nth level? (2 points)

2. A valence electron in an alkali atom is in a p-orbital (*l*=1). Consider the simultaneous interactions of an external magnetic field B and the spin-orbit interaction. The two interactions are described by the potential energy

   $$V = \frac{1}{\hbar^2} A\vec{L}\cdot\vec{S} - \frac{1}{\hbar}\mu_B(\vec{L}+2\vec{S})\cdot\vec{B}$$

   (a) Describe the energy levels for B = 0. (10 points)

   (b) Describe the energy levels for weak magnetic fields (Zeeman effect). What are the Landé g - factors? (10 points)

3. The unit vector $\hat{n}$ is specified by the polar angle θ and azimuthal angle ϕ in the spherical coordinates. $\left|\psi\right\rangle$ is an eigenstate of the operator $\vec{S}\cdot\hat{n}$:

   $$\vec{S}\cdot\hat{n}\left|\psi\right\rangle = (\frac{\hbar}{2})\left|\psi\right\rangle$$

   Show that

   $$\left|\psi\right\rangle = \cos(\frac{\theta}{2})\left|+\right\rangle + \sin(\frac{\theta}{2})e^{i\phi}\left|-\right\rangle,$$

   Where $\left|+\right\rangle$ and $\left|-\right\rangle$ are eigenstates of S_z. (20 points)

Quantum Mechanics Part III (50 points) 2019/3/28

1. [20 points] A particle of mass *m* moves in a one-dimensional potential box: $V(x)=\begin{cases}\infty, & |x|>3|a| \\ 0, & -3a<x<-a \\ V_0, & -a<x<a \\ 0, & a<x<3a\end{cases}$. Consider the constant V_0 part as a perturbation on a flat box of length 6*a*.

   (a) Write down the unperturbed eigenfunctions $\Psi_n^0(x)$ and the corresponding energy eigenvalues E_n^0. (10 points)

   (b) Use the first order perturbation method to calculate the energy of the ground state. (10 points)

2. [30 points] A particle of mass *m* moves in one dimension with the potential $V=g|x|$.

   (a) Calculate the real and positive normalization constant *c* of the function $\psi(x)=c\,\Theta(x+a)\Theta(a-x)\left(1-\frac{|x|}{a}\right)$, where $\Theta(x)=\begin{cases}0, & x<0 \\ 1 & x\geq 0\end{cases}$. (5 points)

   (b) Use the function $\psi(x)$ in (a) as the trial wave function to find the bound of the ground state energy by the variational method. (10 points)

   (c) Minimize the result of (b) to estimate the ground state energy. (5 points)

   (d) Use the WKB method to estimate the ground state energy. (10 points)

Quantum Mechanics 2018/10/18

Part I (50points)

1. (25 Points) For the given matrix $\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

(a) determine the *eigenvalues* and corresponding *orthonormal eigenvectors*.

(b) Find the *unitary transformation matrix* $\hat{S}$ to diagonalize the matrix $\hat{A}$.

(c) What is the finally diagonal matrix $\hat{A}'$?

2. (25 Points) Apply the variational method to determine the *ground state energy* of the 1-D harmonic oscillator. This is the so-called Rayleigh-Ritz variational method which is the minimizing the functional. The Hamiltonian of the 1-D quantum harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

Using the trial function of the form

$$\psi(x) = Ne^{-bx^2}, \qquad b > 0.$$

*Useful integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}; \quad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

Part II (50 points)   2018/10/18

1. $\vec{J} = (J_x, J_y, J_z)$ is the angular momentum operator. (a) Evaluate $\langle J_x^2 \rangle$ in the basis of $|j,m\rangle$. (b) Consider a system with j=1. Explicitly write $\langle j=1, m'|J_y|j=1, m\rangle$ in 3x3 matrix form. (20 points)

2. Consider a 25-25-50 mixture of three spin-1/2 ensembles, with S_z+ (25%), S_x+ (50%), and S_y+ (25%), respectively. (a) Construct the corresponding density matrix of ρ. (b) Evaluate Tr(ρS_x). (10 points)

3. $\sigma_x^{(i)}, \sigma_y^{(i)}$, and $\sigma_z^{(i)}$ are Pauli matrices operating in the space V_i, where i=1,2. The space $V_1 \otimes V_2$ is spanned by four vectors $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$, and $|-\rangle \otimes |-\rangle$. Compute $\sigma_x^{(1)} \otimes \sigma_y^{(2)}$ in the form of 4×4 matrix. (10 points)

4. Consider the problem of two 1/2 spins. To solve the problem, write down the appropriate basis for the following situation and explain why: The two spins are mutually interacting, and the Hamiltonian is written as $H = -A\vec{S}_1 \cdot \vec{S}_2$. (10%)

Part III: 50 %

1. [20%] A system is described by the Hamiltonian $\mathsf{H}_0 + \mathsf{V}$. It is known that the unperturbed system has degenerate states:

$$\mathsf{H}_0|1\rangle = \epsilon_0|1\rangle\ , \mathsf{H}_0|2\rangle = \epsilon_0|2\rangle\ .$$

By using the degenerate perturbation theory, we find that

$$\langle 2|\mathsf{V}|1\rangle = \langle 1|\mathsf{V}|2\rangle = -v < 0\ .$$

Calculate the eigenfunctions and their corresponding eigen-energies of the perturbed system.

2. [15%] Consider a particle of mass m in the one-dimensional infinite well of width a:

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

By using the states for the particle in the infinite well, calculate the first-order corrections to the energies if the particle is in the following potential

$$V(x) = \begin{cases} V_0 & 0 < x \leq a/2, \\ 0 & a/2 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

3. [15%] Write down the following:

   a. [4%] the *general form* of a scattering wave in a collision process.

   b. [7%] the *probability flux density* for a particle of mass m for the scattered wave far from the scatterer.

   c. [4%] the general differential cross section for an *elastic scattering.*

Ph.D. Qualifying Exam, March 22, 2018 (Quantum Mechanics)

Part I: 50 %

1. [6%] Consider the double-slit experiment in FIG. 1 with electrons passing through either slit.

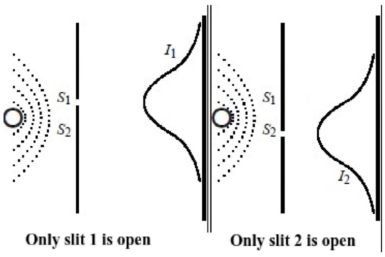


FIG. 1. The double-slit experiment with electrons.

When both slits are open:

(a) [3%] Sketch the intensity pattern if the electrons were classical particles.

(b) [3%] Sketch the actual observed intensity pattern.

2. [24%] Hellmann-Feynman theorem

(a) [7%] Given $\hat{H}(\lambda)|\psi_\lambda\rangle = E_\lambda|\psi_\lambda\rangle$, prove the Hellmann-Feynman theorem:

$$\frac{dE_\lambda}{d\lambda} = \left\langle \psi_\lambda \left| \frac{d\hat{H}}{d\lambda} \right| \psi_\lambda \right\rangle ,$$

where λ is a system parameter.

(b) [7%] For the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi_\lambda(t)}{\partial t} = H_\lambda \Psi_\lambda(t) ,$$

show that we have the following identity instead

$$\left\langle \Psi_\lambda(t) \left| \frac{\partial H_\lambda}{\partial \lambda} \right| \Psi_\lambda(t) \right\rangle = i\hbar \frac{\partial}{\partial t} \left\langle \Psi_\lambda(t) \left| \frac{\partial \Psi_\lambda(t)}{\partial \lambda} \right. \right\rangle .$$

(c) [10%] Consider the Hamiltonian for the quantum harmonic oscillator (QHO) and eigen energies

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 , \ E_n = \left(n + \frac{1}{2}\right) \hbar\omega , \ n = 0, 1, 2, \cdots ,$$

calculate the average kinetic energy $\langle \hat{T} \rangle$ and average potential energy $\langle \hat{V} \rangle$ of the QHO.

3. [20%] Time-dependent system

(a) [10%] Given the wave function $\psi(x,t)$ of a bound particle, show that, for any time t,

$$\frac{d}{dt} \int_{-\infty}^{\infty} \psi(x,t)^* \psi(x,t) dx = 0 .$$

(b) [10%] If the particle is in a stationary state $\psi(x, t_0)$ at time t_0, show that at any later time t, the particle remains in that stationary state.

Quantum Mechanics Qualify Exam Part Ⅱ (50 points) 2018/03/22

1. Suppose a spin-1/2 particle is in the state of $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

   (a) What are the probabilities of getting $+\hbar/2$ and $-\hbar/2$, if you measure S_z and S_x? (10 points)

   (b) Calculate the expectation values of S_z and S_x. (10 points)

2. An electron is at rest in an oscillating magnetic field $\vec{B} = B_0 \cos(\omega t)\hat{k}$, where B_0 and ω are constants.

   (a) The electron starts out (at t = 0) in the spin-up state with respect to the x-axis (i.e., $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time. (10 points)

   (b) Find the probability of getting $-\hbar/2$, if you measure S_x. (10 points)

   (c) What is the minimum magnetic field (B_0) required to force a complete flip in S_x? (10 points)

Quantum Mechanics Qualify Exam Part III (50 points)   2018/3/22

1. Consider a particle moving in an infinite 1-D square well potential ($-a<x<a$). Use the trial wavefunction, $\psi(x)=\begin{cases} N(a^2-x^2)(a^2-\lambda x^2), & |x|<a \\ 0, & |x|\ge a \end{cases}$, to variationally calculate the ground state energy. (N is the normalization constant and λ is the variational parameter). (15 points)
2. Give an example to illustrate/demonstrate the degenerate perturbation method. (15 points)
3. Under the 1st Born approximation and considering elastic scattering, calculate the scattering amplitude, $f(\theta)$, for $V(r)=-\dfrac{e^{-r/b}}{r}$. (10 points)
4. Consider the scattering problem of two identical electrons. What is the differential cross section, $\sigma(\theta)$, if there is no preferred configuration of the total spin? [Assuming that the scattering amplitude of single electron is $f(\theta)$.] (10 points)

Quantum Mechanics 2017/10/19

Part I (50 points)

1. Consider the wave function $\psi(r) = Ne^{-ar}$ in three dimension, where N is a normalization factor and a is a known parameter.
   (a) Calculate the normalization factor N. (5 points)
   (b) Calculate the expectation values $\langle r \rangle$ and $\langle r^2 \rangle$ in this state. (10 points)
   (c) Calculate the uncertainty Δr. (5 points)
   (d) Calculate the probability of finding the particle in the region $r > \Delta r$. (5 points)

2. Consider the one-dimensional potential $V = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x \le a \\ V_0 & \text{for } x > a \end{cases}$, where

   V_0 is positive, and a particle of mass m is in a bound state.

   (a) Calculate the wave function in the region $0 \le x \le a$ without solving the normalization factor. (5 points).
   (b) Calculate the wave function in the region $x > a$ without solving the normalization factor. (5 points)
   (c) Using the conditions satisfied by the wave function at $x = a$, find the equation for the bound state energy. (10 points)
   (d) Find the minimum of V_0 for a bound state to exist. (5 points)

Part II (50 points)

(a) Prove that the *orbital* angular momentum of a particle satisfies the angular momentum algebra. (8 points)

(b) Compute all the basis states and Clebsch-Gordan coefficients of the total system $\vec{J} = \vec{J}_1 + \vec{J}_2$ for two spin 1 particles i.e. for $1 \otimes 1$. (18 points)

(c) Continuing from part **(b)**, what are the eigenvalues and eigenstates of the operator $F = a(J_{1z} + J_{2z}) + b\vec{J}_1 \cdot \vec{J}_2$? Prove your answer. (12 points)

(d) A composite system consisting of two spin $\frac{1}{2}$ particles is assumed to be in the state $\frac{1}{\sqrt{5}}|+\frac{1}{2}, -\frac{1}{2}\rangle + \frac{2i}{\sqrt{5}}|-\frac{1}{2}, +\frac{1}{2}\rangle$ in the product spin basis.
When the *total* spin angular momentum of the system and its z-component are measured, what are the possible values obtained and their respective probabilities?
Derive your answers. (12 points)

Note: You may find the formula $J_{\pm}|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$ useful.

Part III: 50 %

1. [30%] Consider a particle moving in one dimension under the influence of a potential $\lambda V(x)$ with $V(x) = 0$ in the region $|x| > a > 0$, a = constant. The Hamiltonian has the form

$$H = \frac{p^2}{2m} + \lambda V(x) .$$

   (a) [15%] Use the variational principle to show that if $\int V(x)dx < 0$ the Hamiltonian given above has a bound state for an arbitrary small but positive coupling constant λ.

   (b) [15%] Give an upper bound for the energy of this bound state for $\lambda \ll 1$.

2. [10%] The Schrödinger equation for a particle of mass m scattered off a central potential $V(r)$ is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r}) .$$

   If $\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \psi_s(\mathbf{r})$, where

$$\psi_i(\mathbf{r}) = Ae^{i\mathbf{k}\cdot\mathbf{r}} \quad (A = \text{constant}) ,$$

   is the incident wave and

$$\psi_s(\mathbf{r}) = Bf(\theta, \phi)\frac{e^{ikr}}{r} \quad (B = \text{constant}) ,$$

   the scattered wave with $f(\theta, \phi)$ being the so-called scattering amplitude. Find:

   (a) [5%] the incident probability flux $\mathbf{J}_i$,

   (b) [5%] the scattered probability flux $\mathbf{J}_s$.

3. [10%] Obtain the differential scattering cross section in the first Born approximation for the spherically-symmetric potential $V = -V_0 e^{-\alpha r}$ $(V_0 > 0, \alpha > 0)$ in terms of the momentum transfer q and the scattering angle θ. [You may find this integral useful: $\int_0^\infty dx x^n e^{-Ax} = n!/A^{n+1}$.]

Quantum Mechanics Qualify Exam Part I (50 points)   2017/3/23

1. Consider the system in the state described by the wave function $\Psi(x) = C_1\psi(x) + C_2\psi^*(x)$, where C_1 and C_2 are known complex numbers. $\psi(x)$ is a normalized wave function and $\int_{-\infty}^{+\infty}\psi^2(x)dx = D$ is known.
   (a) Write down the normalization condition of Ψ in terms of C_1, C_2 and D. (5 points)
   (b) Using the polar relation $\psi(x) = f(x)e^{i\theta(x)}$, calculate the probability current density $J(x)$ for the state $\Psi(x)$. (10 points)
   (c) Show that the expectation value of the momentum in the state $\Psi(x)$ is given by $<\Psi|p|\Psi> = m\int_{-\infty}^{+\infty}J(x)dx$. (5 points)
   (d) Show that both the probability current density and the expectation value of the momentum vanish if $|C_1| = |C_2|$. (5 points)

2. For the harmonic oscillator, the normalized eigenfunction is given by $<y|n> = A_n e^{-y^2/2}H_n(y)$, where $A_n = \pi^{-1/4}2^{-n/2}(n!)^{-1/2}$ is the normalization constant and $\int_{-\infty}^{+\infty}H_n(y)H_{n'}(y)e^{-y^2}dy = \left(\pi^{1/2}2^n n!\right)\delta_{nn'}$. The lowering and raising operators can be expressed as $a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)$ and $a^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$, such that $a|n> = \sqrt{n}|n-1>$ and $a^\dagger|n> = \sqrt{n+1}|n+1>$.
   (a) Calculate the commutation relation $\left[a, a^\dagger a\right]$. (5 points).
   (b) Using the equation for the lowering operator, derive the recursion relation $H'_n(y) = 2nH_{n-1}(y)$. (5 points)
   (c) Starting with $a + a^\dagger = \sqrt{2}y$, derive the recursion relation $H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y)$. (5 points)

(d) Calculate the integral $\int_{-\infty}^{+\infty} ye^{-y^2} H_n(y) H_m(y) dy$ on which the transition probability between two oscillator states *m* and *n* depends. (5 points)

(e) Calculate the integral $\int_{-\infty}^{+\infty} y^2 e^{-y^2} H_n(y) H_n(y) dy$ which occurs in the calculation of the mean-square displacement of the quantum oscillator. (5 points)

Part II (50%)

Q 1. [Total 26%] Consider four Hermitian 2×2 matrices $\mathbb{I}$, σ_1, σ_2 and σ_3, where $\mathbb{I}$ is the unit matrix, while the others satisfy the anti-commutation relations $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. Without assuming the each σ_i is a Pauli matrix, nor using any specific representation or form for the matrices, show that

(a) [5%] for $i \in \{1, 2, 3\}$, the matrix σ_i is traceless, i.e., $\text{tr}(\sigma_i) = 0$;

(b) [6%] the eigenvalues of σ_i are ± 1 and that $\det(\sigma_i) = -1$;

(c) [7%] the four matrices $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ are linearly independent and therefore that any 2×2 Hermitian matrix can be expanded in terms of them.

(d) [8%] From (c) we know that an arbitrary 2×2 Hermitian matrix M can be decomposed as:

$$M = m_0 \mathbb{I} + \sum_{i=1}^{3} m_i \sigma_i.$$

Derive an expression for m_i in terms of M and σ_i for all $i \in \{1, 2, 3\}$.
[**Hint:** Use the cyclic property of trace and the given anti-commutation relations]

Q 2. [Total 7%] The z-component of the spin of an electron in free space (no electromagnetic fields) is measured and found to be $+\frac{\hbar}{2}$. If the axis defining the measured spin direction $\mathbf{n}$ makes an angle θ with respect to the (original) z-axis, what is the probability of the various possible results? Express your answer in terms of the angle θ.

Q 3. [Total 17%] Consider an electron placed in a uniform magnetic field along the z direction. Suppose that a spin measurement reveals that the electron spin is along the positive y direction at $t = 0$. Find the state vector for the spin, and the average polarization (expectation value of s_x) along the x direction for $t > 0$. Express your answer in terms of $\omega = \frac{\mu_e B}{\hbar}$ where μ_e is the Bohr magneton and B is the magnetic field strength.

Quantum Mechanics Qualify Examine Part III (50 points)

2017/3/23

1. Hamiltonian for a hydrogen-like atom:

$$H = \frac{P^2}{2m} - \frac{Ze^2}{r^2}$$

Due to the spherical symmetry of *H*, the wave function can be decomposed into radial part and angular part:

$$\psi_{nlm}(r\theta\phi) = f_l(r)\, y_{lm}(\theta\phi)$$

where the angular part $y_{lm}(\theta\phi)$ is the spherical harmonic function.

(a) Write the Schrödinger equation for the radial function $f_l(r)$. (5%)

(b) Using variation function:

$$f_l(r) = N r^l\, e^{-ar}$$

where N is the normalization constant; and *a* the variational parameter. Please find *a* to minimize the energy expectation value, and compare the variational energy with the exact eigen energy of hydrogen-like atom for each l. (20%)

2. A quantum particle of mass *m* moves in 1-d periodic potential $V(x) = V_o cos\left(\frac{2\pi x}{a}\right)$. Its time-independent wave function $\psi(x)$, and eigen energy E:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

(a) Show that $\psi(x)$ can differ only a phase factor when shift the position a distance a:

$$\psi(x+a) = e^{i\theta}\psi(x)$$

Wave function is thus classified with angle θ: $\psi(x) \rightarrow \psi_\theta(x)$. (10%)

hint: examining the Fourier expansion of $\psi(x)$.

(b) In case $V_o = 0$, clearly $\psi(x) = e^{ikx}$. We can still classify the wave function with angle θ. What is the eigen energy for $\theta = \pm\pi$ for $V_o = 0$? ($\psi_\pi(x)$ and $\psi_{-\pi}(x)$ are two degenerate states.) (5%)

(c) When V_o is small $\left(\text{i.e. } V_o \ll \frac{\hbar^2}{ma^2}\right)$, calculate the splitting of degenerate states of $\theta = \pm\pi$ using first order perturbation theory. (10%)

PhD Qualify Examine - Quantum Mechanics Part-I

Oct 20, 2016

1. For each of the following time-independent Schrödinger equation, find your best set of length unit a_o and energy unit E_o: $x = a_o\,\xi$ and $E = E_o\,\epsilon$ to scale each equation into a dimensionless and simplest form:

   (a) $-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) - \frac{\hbar^2}{mr^2}R(r) - \frac{1}{2}m\,\omega_o{}^2 r^2 R(r) = E\,R(r)$, (5%)

   (b) $-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) - \frac{3\hbar^2}{mr^2}R(r) - \frac{e^2}{r}\psi R(r) = E\,R(r)$, (5%)

   (c) $-c^2\hbar^2\frac{d^2}{dx^2}\psi(x) + m^2c^4\,\psi(x) = E^2\,\psi(x)$, where c is the speed of light. (5%)

2. For a 1-d simple harmonic oscillator of Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ with commutate relation $[x,p] = i\hbar$, define the rising and lowering operators:

   $$A = \sqrt{\frac{m\omega}{2\hbar}}x + i\sqrt{\frac{1}{2m\hbar\omega}}p, \quad A^+ = \sqrt{\frac{m\omega}{2\hbar}}x - i\sqrt{\frac{1}{2m\hbar\omega}}p$$

   (a) Calculate the commutator $[A,\ A^\dagger]$. (5%)
   (b) Express the Hamiltonian in term of rising and lowering operators. (5%)
   (c) Assume that there exists normalized eigen vector $|\lambda\rangle$ of operator $N = A^\dagger A$ with a eigen value λ, i.e. $N|\lambda\rangle = \lambda\,|\lambda\rangle$. Show that $A|\lambda\rangle$ and $A^\dagger|\lambda\rangle$ are eigen vectors of N with eigen values $\lambda - 1$ and $\lambda + 1$: i.e. $N\,A|\lambda\rangle = (\lambda - 1)\,A|\lambda\rangle$ and $N\,A^\dagger|\lambda\rangle = (\lambda + 1)\,A^\dagger|\lambda\rangle$. (5%)
   (d) Please argue that λ must be a non-negative integer n, $n = 0, 1, 2, 3, \ldots$. (5%)

3. Operator $A(t)$ acts on state $|\psi(t)\rangle$ of a system defined by 1-dimension Hamiltonian $H = \frac{p^2}{2m} + V(x)$:

   $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$; where p is the momentum operator and $V(x)$ the potential energy operator.

   Commutation relation between position x and p is $[x,p] = i\hbar$. The expectation value of $A(t)$ is defined as $\langle A(t)\rangle = \langle\psi(t)|A(t)|\psi(t)\rangle$, where $\langle\psi(t)|$ is the adjoint vector of $|\psi(t)\rangle$, $\langle\psi(t)| = |\psi(t)\rangle^\dagger$

   (a) Prove that Hamiltonian dynamic equation $\frac{d\langle A\rangle}{dt} = \left\langle\frac{\partial A}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[H,A]\rangle$. (5%)

   (b) Using the relation in (a) to show the corresponding theorem: (10%)

   $$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}, \text{ and } \frac{d\langle p\rangle}{dt} = \frac{d\langle V\rangle}{dx}.$$

   If you prefer working in r-representation wave function $\Psi(x,t) = \langle x|\psi(t)\rangle$, the expectation value then becomes an integral:

   $$\langle A(t)\rangle = \int_{-\infty}^{\infty}\Psi(x,t)^*A(t)\Psi(x,t)\,dx\,.$$

Quantum Mechanics Part II (50 points)

1. (12 points) Suppose that we have a particle of mass m constrained to move on a circle of radius a. What is the Hamiltonian of such a particle? Please also calculate its eigenvectors and eigenenergies, and interpret the degeneracy.

2. (12 points) Consider a system of two spin-1/2 particles (whose orbital degrees of freedom we ignore). Let particle 1 in the $S_z = +\hbar/2$ eigenstate and particle 2 in the $S_x = +\hbar/2$ eigenstate. What is the probability that a measurement of the total spin will give the value zero?

3. (26 points) Consider a spin-1/2 particle being placed in a magnetic field $\mathbf{B} = B_0\hat{z} + B_1\cos\omega t\hat{x} + B_1\sin\omega t\hat{y}$, which is often employed in magnetic resonance experiments. Assume that the particle has spin up along the $+z$-axis at $t = 0$ ($m_z = +1/2$). Please derive the probability to find the particle with spin down ($m_z = -1/2$) at time $t > 0$.

Quantum Mechanics Qualify Exam (Part III) 2016/10/20

1. Imagine two non-interacting identical fermions, each of mass m, in the infinite square well. If one is in the state ψ_n (where $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$), and the other in state ψ_l $(l \neq n)$, calculate $\langle (x_1 - x_2)^2 \rangle$. (15%)

2. Consider two identical harmonic oscillators of mass m and spring constant k. The interacting potential is given by $H_I = c\, x_1 x_2$ where x_1 and x_2 are the oscillator variables.
   (a) Find the exact energy levels of the total Hamiltonian. (6%)
   (b) Assume $c << k$ and compute the lowest pair of excited states in first-order perturbation theory. (Give energy in first order and eigenfunctions in zeroth order.) (14%)

3. Let $\vec{k}$ and $\vec{k'}$ be respectively the wave vectors of the incident and scattered waves. Using Born approximation to calculate the differential and total cross section for scattering a particle of mass m off the δ-function potential $V(\vec{r}) = g\,\delta^3(\vec{r})$. (15%)

Quantum Mechanics (Part I, 50 points) 2016-03-24

1. (a) The Bohr's model of the hydrogen atom is based upon several assumptions. Discuss these assumptions and their significances; also describe them in math equations. **(10 points)**

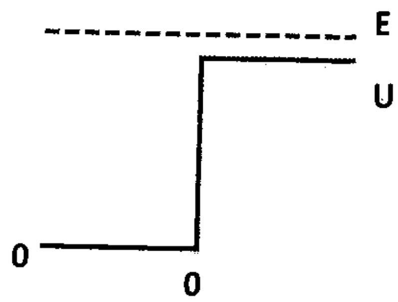
   (b) The spectral lines corresponding to values of quantum number n from 3 to 6 down to 2 are in the visible range of the electromagnetic spectrum, called the Balmer series. Following your assumptions in (a), please show the **wavelengths** of the spectral lines of Balmer series. **(5 points)**

   (c) Please write down two postulates of Einstein's principle of relativity. **(5 points)**

   (d) When an atom makes a transition between states, energy is emitted in the form of a photon. Although an excited atom can radiate at any time from $t = 0$ to $t = \infty$, the average time interval after excitation during which an atom radiates is called the lifetime τ. If $\tau = 1.8 \times 10^{[\text{illegible}]}$ s, use the uncertainty principle to compute the line width produced by the finite lifetime. **(5 points)**

2. An alien spaceship traveling at 0.600 c toward the Earth launches a landing craft. The landing craft travels in the same direction with a speed of 0.800 c relative to the mother ship. As measured on the earth, the spaceship is 0.200 ly (light year, length unit) from the earth when the landing craft is launched. (a) What speed do the earth-based observers measure for the approaching landing craft? (b) What is the distance to the Earth at the moment of the landing craft's launch as measured by the aliens? (c) What travel time is required for the landing craft to reach the Earth as measured by the aliens on the mother ship? (d) If the landing craft has a mass of $4.00 \times 10^{[\text{illegible}]}$ kg, what is its kinetic energy as measured in the Earth reference frame? **(10 points)**

3. Consider the step potential in the case where E > U as shown in right figure. (a) Examine the Schrödinger equation to the left of the step to find the form of the solution in the range x < 0. Do the same to the right of the step to obtain the solution form for x > 0. Complete the solution by enforcing whatever boundary and matching conditions may be necessary. (b) Obtain an expression for the reflection coefficient R in the case, and show that it can be written in the form $R = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$, where k_1 and k_2 are wavenumbers for the incident and transmitted waves, respectively. Also write an expression for the transmission factor T using the sum rule obeyed by these coefficients. (c) Evaluate R and T in the limiting cases of $E \to U$ and $E \to \infty$. Are the results sensible? Please explain. **(15 points)**



Note:

1. Plank's constant $h = 6.626 \times 10^{-34}$ J·s
2. Mass of electron = 9.11×10^{-31} kg
3. 1 eV = 1.602×10^{-19} J
4. Boltzmann constant $k_b = 1.380 \times 10^{-23}$ J·K^{-1}
5. 1 u = 931.494013 MeV/c^2 = $1.66053873 \times 10^{-27}$ kg

Quantum Mechanics Part II (50 points)

1. [15 points] As we know that parity is a unitary operator, show and explain why the time-reversal operator has to be antiunitary.

2. [15 points] The Schrodinger's equation says that the dynamical evolution of $\Psi(t,\vec{x})$ is given by

$$i\hbar\frac{\partial\Psi(t,\vec{x})}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi(t,\vec{x}) + V(\vec{x})\Psi(t,\vec{x}).$$

From the Schrodinger's equation, show that the probability conservation is satisfied and define the corresponding probability density and current density, i.e., $\partial_\mu J^\mu = \frac{\partial\rho}{\partial t} - \vec{\nabla}\cdot\vec{J} = 0.$

3. [20 points] It is known that the commutation relations of orbital angular momentum are

$$[\hat{L}_i, \hat{L}_j] = i\hbar\,\epsilon_{ijk}\hat{L}_k.$$

The eigenvalue equation of $\hat{L}^2 = \sum_i \hat{L}_i^2$ is defined as $\hat{L}^2|\,\ell,m> = \ell(\ell+1)\hbar^2|\ell,m>$, where ℓ is the quantum number of angular momentum and m is the eigenvalue of $\hat{L}_z$. Find the representations of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in the angular momentum state with $\ell = 1$.

Part III: 50 %

1. [5%] The scattering amplitude of a scattering process has the form

$$f(\theta, \phi) = -b_1 + ib_2\,, \; b_1 \text{ and } b_2 \text{ are real constants.}$$

Find the total cross-section σ_{total}.

2. [15%] Let the groundstate eigen-energy of a quantum system be E_0 with corresponding eigenfunction $\psi(\vec{x})$: $\hat{\mathscr{H}}\psi(\vec{x}) = E_0\psi(\vec{x})$. For any trial wave function $\phi(\vec{x})$, show that

$$\frac{\int \phi^*(\vec{x})\hat{\mathscr{H}}\phi(\vec{x})d^3\vec{x}}{\int \phi^*(\vec{x})\phi(\vec{x})d^3\vec{x}} \geq E_0.$$

The equality is satisfied if $\phi(\vec{x}) = \psi(\vec{x})$.

3. [10%] Define the wave function in the interaction picture by $|\phi(t)\rangle_I$. It is related to the wave function in the Schrödinger picture by

$$|\psi(t)\rangle_S = e^{-iH_0t/\hbar}|\psi(t)\rangle_I\ .$$

   (a) Find the time-dependent equation in the interaction picture.

   (b) Let $V_I(t)$ be the potential in the interaction picture. Find its relation with the potential V_S in the Schrödinger picture.

4. [20%] Consider a system with a simple 2×2 matrix Hamiltonian $H = H_0 + V$ with

$$H_0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix},$$

where v is a complex constant.

   (a) Find the exact energies of the system.

   (b) Consider V as a perturbation to the unperturbed system H_0. Use perturbation theory to find the energies for the non-degenerate and degenerate cases: $E_1^0 \neq E_2^0$ and $E_1^0 = E_2^0$.

Quantum Mechanics (Part I, 50 points) 2015-10-15

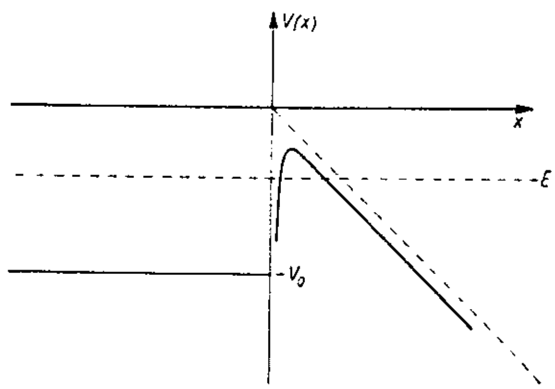
1. (a) What assumptions did Plank make in dealing with the problem of blackbody radiation? Discuss the consequences of these assumptions. **(5 points)**

   (b) In the photoelectric effect, explain why the stopping potential depends on the frequency of light but not on the intensity. **(5 points)**

   (c) The wavefunction of a free electron is $\Psi(x) = A\sin(5\times10^{10}x)$, where x is in meters. What is the momentum of the electron? **(5 points)**

   (d) A particle's wave function is $\psi = Ae^{-x^2/a^2}$, where A and a are constants. (a) Where is the particle most likely to be found? (b) Where is the probability per unit length half its maximum value? **(10 points)**

2. In reality, the change in potential near the surface of a metal is continuous. Thus, e.g., the potential of an electric image $V_{e.i.} = -\frac{e}{4x}$ acts over a large distance from the surface. Taking into account the force due to the electric image, determine the transmission coefficient D through the surface of a metal in an electric field. **(15 points)**



3. Find the wave functions of a charged particle in a uniform field $V(x) = -Fx$. **(10 points)**

Note:

1. Plank's constant $h = 6.626 \times 10^{-34}$ J·s

2. Mass of electron = 9.11×10^{-31} kg

3. 1 eV = 1.602×10^{-19} J

4. Boltzmann constant $k_b = 1.380 \times 10^{-23}$ J·K^{-1}

5. The Airy function $\Phi(q) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos(\frac{u^3}{3} + uq) du$

Quantum Mechanics Part II (50 points) 15/Oct/2015

1. Consider the three-dimensional harmonic oscillator, for which the potential is

$$V(r) = \frac{1}{2}m\omega^2 r^2.$$

[10 points] (a) Show the allowed energies of the quantum system E_n.

[5 points] (b) Explain what the stationary state means in quantum mechanics (QM).

[10 points](c) Using $\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$ where $\hat{H}$ is Hamiltanian, show that in stationary state, the virial theorem to the three-dimensional harmonic oscillator is

$$\langle T\rangle = \langle V\rangle = \frac{E_n}{2}.$$

2. [5 points] (i) Select and explain the correct statement: Heisenberg's uncertainty principle is a (A) postulate in QM, (B) result of QM.

[5 points] (ii) How to describe the situation in QM when a system has a symmetry under the operation of $\hat{O}$?

3. [15 points] The equation of motion for a charged particle moving in a constant magnetic field **B** can be obtained by shifting the phase of wave function, $\Psi \to e^{ig}\Psi$ in which the wave function Ψ obeys

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.$$

If the phase factor is given by $g(\boldsymbol{r}) = \frac{q}{\hbar}\int^{r} A(\boldsymbol{r}')\cdot d\boldsymbol{r}'$, show that the phase difference for a closed contour, which encircles a long solenoid with radius a, is $\frac{q\Phi}{\hbar}$ with $\Phi = \pi a^2 B$ being the magnetic flux.

Quantum Mechanics Qualify Exam Part III (50 points) 2014/10/15

1. Consider a particle of mass m and charge e in the central potential $V(r)=\begin{cases} -\dfrac{e^2}{r}, & 0<r<R \\ -\dfrac{e^2}{r}\exp[-\lambda(r-R)], & R<r<\infty \end{cases}$. This potential differs from the Coulomb potential only in the region $r > R$, where the Coulomb force is screened.

   (a) Write down the Hamiltonian as an unperturbed Hamiltonian with the Coulomb potential plus the potential difference ΔV. (5 points)

   (b) Consider the difference in (a) as a perturbation and calculate the first-order correction to the energy of the ground state. (15 points)

   (c) Check the result in (b) in the limit $\lambda \to 0$ and in the limit $R \to \infty$ separately. (5 points)

   [Hint: unperturbed ground state $\psi_{100} = \left(\pi a_0^3\right)^{-1/2} \exp\left(-r/a_0\right)$.]

2. Consider the one-dimensional potential $V(x)=\dfrac{\lambda x^4}{4}+\dfrac{\lambda a x^3}{4}-\dfrac{\lambda a^2 x^2}{8}$.

   (a) Find the points of classical equilibrium for a particle of mass m moving under the influence of this potential. (10 points)

   (b) Using the variational method, consider the trial wave function $\psi(x)=\left(\dfrac{\beta}{\pi}\right)^{1/4} \exp\left[-\beta\left(x-x_0\right)^2/2\right]$, where x_0 is the global minimum found in (a). Now take a special value of the coupling constant $\lambda=\hbar^2/\left(ma^6\right)$. Evaluate the expectation value of the energy for this wave function and get an estimate of the ground-state energy. (15 points)