Classical Electrodynamics Part I (50 points)

Useful relations:

$\hat{r} = sin\theta cos\phi \hat{x} + sin\theta sin\phi \hat{y} + cos\theta \hat{z}$

$\hat{\theta} = cos\theta cos\phi \hat{x} + cos\theta sin\phi \hat{y} - sin\theta \hat{z}$

$\hat{\phi} = -\sin\phi\,\hat{x} + cos\phi \hat{y}$

$$\int_d^0 \frac{\sqrt{x}}{\sqrt{d-x}}dx = -d\frac{\pi}{2}$$

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{rsin\theta}\frac{\partial}{\partial \theta}(sin\theta V_\theta) + \frac{1}{rsin\theta}\frac{\partial V_\phi}{\partial \phi}$$

1. [15 points] The potential generated by a dipole $\vec{p}$ is given as

$$V_{dip}(r,\theta) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}.$$

Derive the electric field $\vec{E}$ and write it in coordinate-free form.

2. [15 points] A point charge q of mass m is released from a rest at a distance d from an infinite grounded conducting plane. How long will it take for the charge to hit the plane?

3. [20 points] The electric potential of some configuration is given as:

$$V(\vec{r}) = A\frac{\exp[-\lambda r]}{r},$$

where A and λ are constants. Find the electric field $\vec{E}(\vec{r})$, the charge density $\rho(r)$, and the total charge Q.

Classical Electrodynamics

Part II (50 points)

1. The Biot-Savart law reads
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\mathbf{r}}{|\mathbf{r}|^3}\ , \tag{1}$$
where μ_0 is the free-space permeability.
   a. **(5 points)** Explain Eq. (1) briefly with a proper sketch.
   b. **(7 points)** Consider a point charge Q moving at velocity $\mathbf{v}$, as sketched in Fig. 1(a). Find the magnetic field $\mathbf{B}$ at an arbitrary point P located at $\mathbf{r}$ relative to the point charge, using the Biot-Savart law (1). Express your result in terms of the speed of light in vacuum, $c = 1/\sqrt{\mu_0\varepsilon_0}$, where ε_0 is the free-space permittivity, the electrostatic field $\mathbf{E}$ due to the point charge Q, and the velocity $\mathbf{v}$.
   c. **(8 points)** Two point charges Q_1 and Q_2 move side by side at the same velocity in vacuum, as sketched in Fig. 1(b). Apply the result found in 1b to find the magnitude of the force between Q_1 and Q_2.

2. Consider a current density function $\mathbf{J}$ in free space; see Fig. 1(c). Complete the following tasks. Note that "primed" and "unprimed" coordinates and differential operators should be clearly and carefully distinguished.
   a. **(4 points)** Show that the magnetic field at position $\mathbf{r}$ due to $\mathbf{J}$ can be written as
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\ , \tag{2}$$
   based on the Biot-Savart law (1).
   b. **(3 points)** Prove
$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\vec{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$
   and use it to show that Eq. (2) can be rewritten as
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\ . \tag{3}$$
   c. **(3 points)** Show that Eq. (3) is divergenceless, i.e., $\vec{\nabla}\cdot\mathbf{B} = 0$.
   d. **(11 points)** Calculate $\vec{\nabla}\times\mathbf{B}$ based on Eq. (3). Hint: There should be two terms left. Furthermore, the following identities may be directly used:
$$\vec{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\vec{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \qquad \nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$
   e. **(3 points)** Write down the equation of continuity, and explain it briefly.
   f. **(2 points)** Apply the equation of continuity at steady-state to the result of 2d.
   g. **(4 points)** Use the result of 2f to derive Ampère's law.

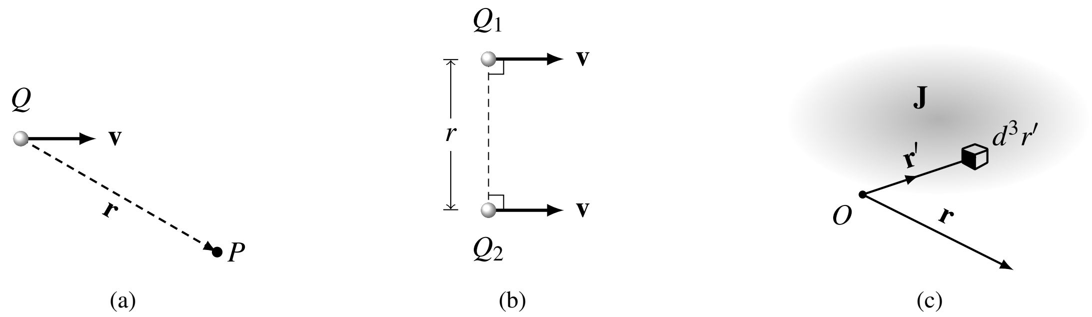


Figure 1: Sketches for (a) problem 1b, (b) problem 1c, and (c) problem 2a.

Qualifying Exam: Electrodynamics
Part 3: 50%

1. [20%] An electric field of an electromagnetic wave in free space has the following form:

$$\mathbf{E}(\mathbf{r},t) = 10(\hat{\mathbf{e}}_x + 2\hat{\mathbf{e}}_y + E_x\hat{\mathbf{e}}_z)\cos(\omega t + 3x - y - z) \quad (\mathrm{V/m})$$

   (a) [10%] Determine the direction of propagation and the values E_x and ω.

   (b) [10%] What is the time-average energy flow?

2. [10%] A plane wave in free space propagating in the $+z$ direction is

$$\mathbf{E} = \hat{\mathbf{e}}_x 100\sin(\omega t - kz) + \hat{\mathbf{e}}_y 200\cos(\omega t - kz) \quad (\mathrm{V/m})$$

   The wave is incident on an infinite conducting xy plane at $z = 0$. What is the reflected wave $\mathbf{E}_r$?

3. [10%] Let θ_i be the angle of incidence of a wave from a medium with index of refraction n to a medium with n'. For the polarization of the wave perpendicular to the plane of incidence, the reflection coefficient R reads:

$$R = \left| \frac{n\cos\theta_i - n'\sqrt{1-(n/n')^2\sin^2\theta_i}}{n\cos\theta_i + n'\sqrt{1-(n/n')^2\sin^2\theta_i}} \right|^2 ,$$

   while for polarization parallel to the plane of incidence, the coefficient reads:

$$R = \left| \frac{n'\cos\theta_i - n\sqrt{1-(n/n')^2\sin^2\theta_i}}{n'\cos\theta_i + n\sqrt{1-(n/n')^2\sin^2\theta_i}} \right|^2 .$$

   (a) [5%] Determine Brewster's angle θ_B for a total transmission of wave energy.

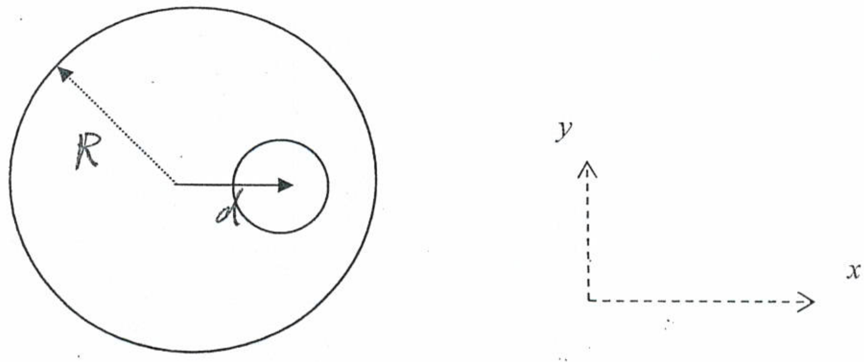
   (b) [5%] Determine critical angle θ_C for a total internal reflection.

4. [10%] Let $\phi(\mathbf{r},t)$ be a scalar potential and $\mathbf{A}(\mathbf{r},t)$ a vector potential.

   (a) [5%] Write down the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic fields $\mathbf{B}(\mathbf{r},t)$ in terms of $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$.

   (b) [5%] Write down the gauge transformation for $\phi(\mathbf{r},t) \longrightarrow \phi'(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t) \longrightarrow \mathbf{A}'(\mathbf{r},t)$.

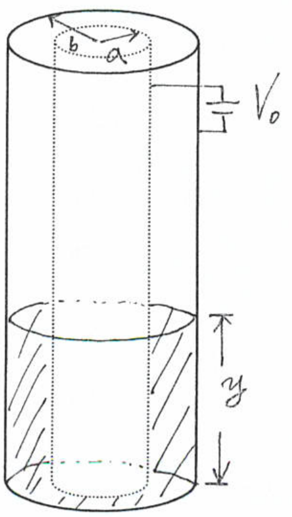
1. **(10 Points)** Find the magnitude and direction of the electric field $\vec{E}$ in the spherical cavity whose center is displaced from that of the solid part by a distance d. The uniform charge density is ρ_o



2. (15 Points) A point charge q is embedded at the center of a sphere of linear dielectric material (electric permittivity ε and radius R). (a) Find the electric displacement $\vec{D}$, the electric field $\vec{E}$, as well as the polarization $\vec{P}$ inside the sphere. (b) What is the polarization surface charge density?

3. (10 Points) An electric dipole $\vec{p}$ with a magnitude of 2×10^{-27} C-m is initially located at an origin along the z-axis. (a) Determine the rotational torque on this dipole when an electric field $\vec{E} = 5\hat{x} + 10\hat{y}$ (V/m) is applied. (b) How much energy changes if $\vec{p}$ finally points to the direction of $\vec{E}$?

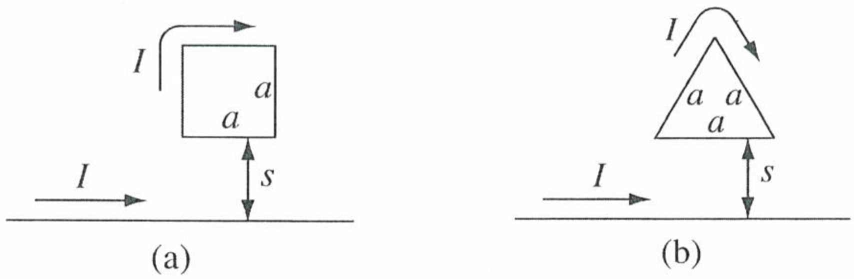
4. (15 Points) A long coaxial cylindrical capacitor of inner radius a and outer radius $b = 2a$ stands vertically in a tank of dielectric oil (electric permittivity ε and mass density ρ_m). The inner one is maintained at potential V_o, while the outer one is grounded. To what height y does the oil rise in the space between a and b.



Classical Electrodynamics Part II (50 points)

1. [10 points] (a) Find the force on a square loop placed as shown in Fig. 1(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

   [10 points] (b) Find the force on the triangular loop in Fig. 1(b).



2. [10 points] What current density would produce the vector potential, $\mathbf{A} = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates?

3. [10 points] (a) Starting from the Lorentz force law, in the form as:

$$\vec{F} = \int I(d\vec{\ell} \times \vec{B}),$$

   Find that the torque on *any* steady current distribution (not just a square loop) in a uniform field is $\vec{m} \times \vec{B}$, where $\vec{m}$ is the magnetic dipole moment of a loop.

   [10 points] (b) A uniform current density $\mathbf{J} = J_0\,\hat{z}$ fills a slab straddling the yz plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0\,\hat{x}$ is situated at the origin. Find the force on the dipole using $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$.

成功大學物理研究所博士班 108 學年度下學期 [資格考試]

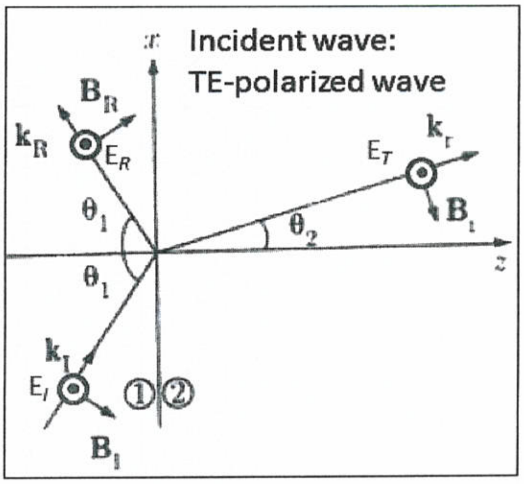
Electrodynamics (Part III, 50 points) 2020-03-27

1. Assuming that a monochromatic point light source is spreading out into every direction.

   (a) Please write down the wave function of the wave and show that your solution satisfies the wave equation. **(15 points)**

   (b) What is the expression for the Poynting vector of the wave? Find the momentum density of this wave **(10 points)**

2. Referring to the figure, an electromagnetic wave of polarization perpendicular to the plane of incidence is obliquely impinging on the boundary. Medium 1 and 2 are both dielectrics. The incident, reflected, transmitted waves are represented as following:



$$\begin{cases} \tilde{\mathbf{E}}_I &= \tilde{E}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I &= \dfrac{1}{v_1} \tilde{E}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} (-\cos\theta_1\, \hat{\mathbf{x}} + \sin\theta_1\, \hat{\mathbf{z}}); \end{cases}$$

$$\begin{cases} \tilde{\mathbf{E}}_R &= \tilde{E}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R &= \dfrac{1}{v_1} \tilde{E}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} (\cos\theta_1\, \hat{\mathbf{x}} + \sin\theta_1\, \hat{\mathbf{z}}); \end{cases}$$

$$\begin{cases} \tilde{\mathbf{E}}_T &= \tilde{E}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T &= \dfrac{1}{v_2} \tilde{E}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} (-\cos\theta_2\, \hat{\mathbf{x}} + \sin\theta_2\, \hat{\mathbf{z}}); \end{cases}$$

   (a) Write down boundary conditions of E-field and B-field. **(10 points)**

   (b) Impose the boundary conditions and obtain the Fresnel equations for E_{0R} and E_{0T}. **(15 points)**

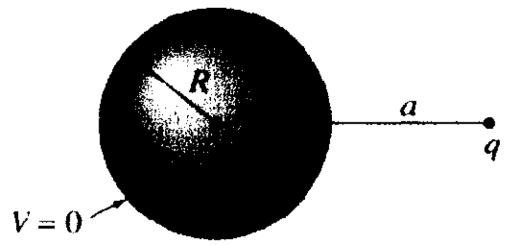
Electrodynamics, Part I (50 points)

1. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R.

   (a) Find the potential outside the sphere. **(10 points)**

   (b) Find the induced surface charge distribution on the sphere, and the total induced charge on the sphere. **(10 points)**

   (c) Calculate the energy of this configuration. **(10 points)**



2. A dielectric sphere having a radius a and a uniform linear dielectric constant ε_r is placed in an otherwise uniform electric field E_0. Find:

   (a) the potential inside the sphere, **(5 points)**

   (b) the potential outside the sphere, **(5 points)**

   (b) the electric field inside the sphere, and **(5 points)**

   (c) the polarization-surface-charge density. **(5 points)**

Classical Electrodynamics Part II (50 points)

1. [10 points] As shown in Fig. 2.1, an infinitely long wire carries a current $I = 1$ A. It is bent so as to have have a semi-circular detour around the origin, with radius 1 cm. Calculate the magnetic field at the origin. [permeability of free space $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$]

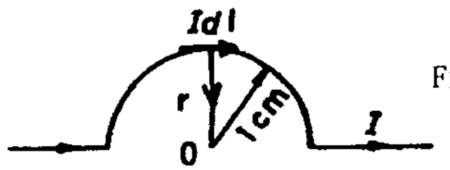


Fig. 2.1

2. [15 points] A cylindrical wire of permeability μ carries a steady current **I**. If the radius of the wire is R, find **B** and **H** inside and outside the wire.

3. [25 points] A long non-magnetic cylindrical conductor with inner radius a and outer radius b carries a current **I**. The current density in the conductor is uniform. Find the magnetic field set up by this current as a function of radius

   (a) inside the hollow space (r < a);

   (b) within the conductor (a < r < b);

   (c) outside the conductor (r > b).

Electrodynamics (Part III 50 points)

1. (a) Describe what is the gauge invariance of $\vec{E}$ and $\vec{B}$ fields. (4 points)

   (b) Describe what is Coulomb gauge and what is Lorentz gauge. (4 points)

   (c) Derive the wave equations for the scalar potential and vector potential when using Lorentz gauge. (8 points)

2. (a) Write down Maxwell's equations in media. (4 points)

   (b) By using Maxwell's equations, derive the wave equations of $\vec{E}$ fields for an electromagnetic wave traveling in vacuum. Also prove that the propagation speed of electromagnetic waves in vacuum is light speed. (6 points)
   (Hint: $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}$).

   (c) Prove that a monochromatic electromagnetic plane wave traveling in vacuum is a TEM wave. (6 points)

   (d) Use Maxwell's equations, derive the boundary conditions for E and B fields of electromagnetic waves across the boundary between two linear media (electric permittivity: ε_1, ε_2, permeability: μ_1, μ_2, respectively). (6 points)

   (e) Derive the dispersion relation $k(\omega)$ for electromagnetic plane waves transferring in a conducting medium. (6 points)

   (f) Give an example: Under what conditions, you can find the propagation of a non-TEM wave (such as TE or TM wave)? (6 points)

Electrodynamics, Part I, 50 points **2018 – 10 – 19**

1. [15 points] Use Gauss's theorem to prove the following:

   (a) Any excess change placed on a conductor must lie entirely on its surface. (A conductor by definition contains changes capable of moving freely under the action of applied electric fields) (5 points)

   (b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it. (5 points)

   (c) An electric charge distribution produces an electric field

$$\vec{E} = A(1 - e^{-\alpha r})\frac{\hat{r}}{r^2}$$

   where A and α are constants. Find the net charge within the radius $r = 1/\alpha$. (5 points)

2. [14 points] An insulated, spherical, conducting shell of radius R is in a uniform electric field $\vec{E}_0$. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

   (a) if the shell is uncharged; (7 points)

   (b) if the total charge on the shell is Q. (7 points)

3. [21 points] An amount of charge q is uniformly spread out in a layer on the surface of a disc of radius a.

   (a) Use elementary methods based on the azimuthal symmetry of the charge distribution to find the potential at any point on the axis of symmetry. (6 points)

   (b) With the aid of (a) find an expression for the potential at any point $\vec{r}$ $(|\vec{r}| > a)$ as an expansion in Legendre polynomials. (15 points)

Classical Electrodynamics Part II (50 points)

1. [20 points] A magnetic induction $\vec{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho,z)$ and $B_\rho(\rho,z)$ and with a known $B_z(0,z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z. Show that near the axis and radial components of magnetic induction are approximately

$$B_z(\rho,z) \approx B_z(0,z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots$$

$$B_\rho(\rho,z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots$$

2. [15 points] A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential both inside and outside the sphere.

Some identities: (i) $\int \frac{\hat{r}'}{|\vec{r}-\vec{r}'|} d\Omega' = f(r)\hat{r}$,

(ii) $\int \frac{\cos\gamma}{|\vec{r}-\vec{r}'|} d\Omega' = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} \int P_1(\cos\gamma)P_\ell(\cos\gamma)\, d\Omega'$, where $\cos\gamma = \hat{r}\cdot\hat{r}'$, $r_< = \min(r,a)$, $r_> = \max(r,a)$, and P_ℓ denote the Legendre polynomials .

3. [15 points] (a) How to define the magnetic dipole moment of a current loop? (use your notations)

   (b) When a magnetic dipole moment $(\vec{m})$ is put in a uniform magnetic induction field $(\vec{B})$, how to derive the potential energy of the system?

Part III: (50 points)

(1) Electrons oscillate inside plasma under the static magnetic induction $\vec{B}_s = B_0\hat{z}$ and a plane EM wave, $\vec{E}(t,\vec{x}) = E_0(\hat{x} \pm i\hat{y})e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Assume that the wave length of the EM wave is much smaller than a given electron's oscillation. You might use following notations (but unnecessarily use each of them) for completing questions: m: electron mass; e: absolute value of electron charge; ω_0: intrinsic angular frequency; γ: damping constant; N: number per volume.

(a) Write down the equation of motion (5pts).

(b) Find a steady-state solution which means "ignoring homogeneous solutions" (10pts).

(c) Compute the polarization (5pts).

(d) Obtain permittivity (2pts).

(e) Is this medium isotropic (1pt)? Give reasons for your answer (2pts).

(2) A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\vec{\nabla}\Phi$. In this region, it also exists a small localized time-independent current density $\vec{J}(\vec{x})$, which generates a magnetic field $\vec{H}$.

(a) Show that the momentum of these EM fields, $\vec{P} \equiv \int d^3x[\vec{D}\times\vec{B}]$, can be transformed to $\vec{P} = \frac{1}{c^2}\int d^3x\, \Phi\vec{J}$ by dropping the surface term (5pts).

- NB: $\epsilon^{ijk}(\frac{\partial}{\partial x^j}A)B^k = \epsilon^{ijk}\frac{\partial}{\partial x^j}[AB^k] - \epsilon^{ijk}A\frac{\partial}{\partial x^j}B^k$

(b) Justify in which situation you can ignore the surface term in part (a) (2pts).

(c) Assuming that the current distribution is localized to a region small compared to the scale of the variation of the electric field, that is, using $\Phi = \Phi_0 - \vec{E}_0\cdot\vec{x}$. Show $\vec{P} = \frac{1}{c^2}\vec{E}_0 \times \vec{m}$, where $\vec{m}$ is a magnetic dipole moment (8pts).

- NB1: localized source $\Longrightarrow \int d^3x J^i(\vec{x}) = 0, \quad \int d^3x[x^k J^\ell(\vec{x}) + x^\ell J^k(\vec{x})] = 0$
- NB2: $\epsilon^{ijk}\epsilon^{k\ell m} = \delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}$

(d) Suppose the current distribution is placed instead in a uniform electric field $\vec{E}_0$ (filling all space). Show that the result in (c) is changed by a surface integral contribution, yielding to $\vec{P} = \frac{2}{3c^2}\vec{E}_0 \times \vec{m}$ (10pts).

- NB: $\int d\Omega\, \hat{r}^i = 0, \qquad \int d\Omega\, \hat{r}^i\hat{r}^j = \frac{4\pi}{3}\delta^{ij}$

Part III: (50 points)

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- NB: $\epsilon^{ijk}(\frac{\partial}{\partial x^j}A)B^k = \epsilon^{ijk}\frac{\partial}{\partial x^j}[AB^k] - \epsilon^{ijk}A\frac{\partial}{\partial x^j}B^k$

(b) Justify in which situation you can ignore the surface term in part (a) (2pts).

(c) Assuming that the current distribution is localized to a region small compared to the scale of the variation of the electric field, that is, using $\Phi = \Phi_0 - \vec{E}_0\cdot\vec{x}$. Show $\vec{P} = \frac{1}{c^2}\vec{E}_0 \times \vec{m}$, where $\vec{m}$ is a magnetic dipole moment (8pts).

- NB1: localized source $\Longrightarrow \int d^3x J^i(\vec{x}) = 0, \quad \int d^3x[x^k J^\ell(\vec{x}) + x^\ell J^k(\vec{x})] = 0$
- NB2: $\epsilon^{ijk}\epsilon^{k\ell m} = \delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}$

(d) Suppose the current distribution is placed instead in a uniform electric field $\vec{E}_0$ (filling all space). Show that the result in (c) is changed by a surface integral contribution, yielding to $\vec{P} = \frac{2}{3c^2}\vec{E}_0 \times \vec{m}$ (10pts).

- NB: $\int d\Omega\, \hat{r}^i = 0, \qquad \int d\Omega\, \hat{r}^i\hat{r}^j = \frac{4\pi}{3}\delta^{ij}$

Electrodynamics (Part I: Electrostatics 50 points).

1. (a) From the Green's theorem (Green's second identity):

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_s [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]da\,,$$

where ϕ and ψ are arbitrary scalar fields, show that the solution of a Dirichlet boundary-value problem is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho_f(\vec{r}\,')G_D(\vec{r},\vec{r}\,')d\tau' - \frac{1}{4\pi}\oint_S \Phi(\vec{r}\,')\frac{\partial G_D}{\partial n'}da'\,.$$

To do so, what are the required conditions to choose a suitable Green's function G_D? **(8 points)**

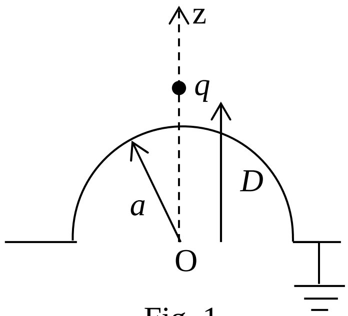


Fig. 1

(b) As shown in Fig. 1, the conducting hemi-spherical shell of radius *a* and the infinite conducting plane are grounded. A point charge q (q>0) is placed on the z axis and the distance from the charge to the center of the hemi sphere is D.

(i) Find the positions and magnitudes of all image charges. **(6 points)**
(ii)Write down the appropriate Green function $G(\vec{r},\vec{r}\,')$ for Dirichlet boundary value problems in the region z>0 and r>a. **(5 points)**

2. As shown in Fig. 2, the surface charge density on a thin spherical shell with radius *b* is $\sigma_f(\theta')=\sigma_f\cos^2\theta'$. Find the electric potential inside ($r<b$) and outside ($r>b$) the spherical shell. **(15 points)**

Legendre polynomials: $P_o(x) = 1; P_1(x) = x;$

$P_2(x) = (3x^2 - 1)/2; P_3(x) = (5x^3 - 3x)/2; \ldots$

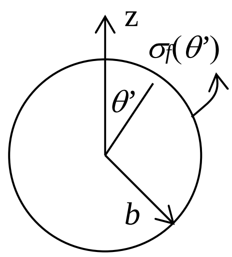


Fig. 2

3. As shown in Fig. 3, charge *Q* was uniformly distributed on a spherical conductor surface (radius *a*). The conductor was surrounded by a dielectric shell of inner radius *a* and outer radius *b*. The electric permittivity of the linear dielectric is ε.

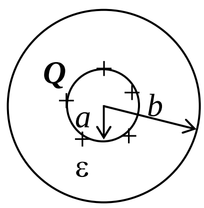


Fig. 3

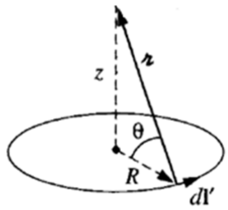
   (a) Find the electric fields $\vec{E}$ at $b>r>a$ and $r>b$. **(6 points)**
   (b) Find the polarization surface bound charges at $r=a$ and $r=b$. **(4 points)**
   (c) Find the electric force per unit volume on the dielectric (Express your answer by using the position vector from the sphere center: $\vec{r}$). **(6 points)**

Part II: Magneto-statics (50 points)

1.

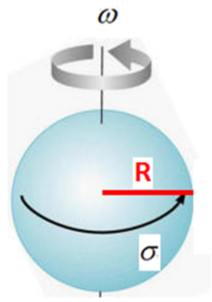
For a line current distribution $\mathbf{I(r')}$, Biot –Savart law can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$$



(1) show that the magnetic field at a distance above the center of a circular loop of radius R, and carries a current I is

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\ \hat{\mathbf{z}}$$

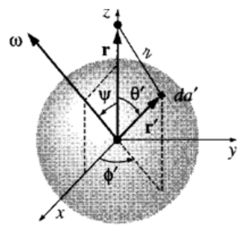


(2)using the results of (1), show that the magnetic field at the center of a sphere of radius R, carrying a uniform surface charge σ and spinning at anglar velocity $\boldsymbol{\omega}$ is

$$B = \frac{2}{3}\mu_0 R\sigma\omega\,\hat{\mathbf{z}} \quad \left(\text{or } \frac{\mu_0 Q\omega}{6\pi R}\,\hat{\mathbf{z}}\,,\ \sigma = \frac{Q}{4\pi R^2}\ \right)$$

2.

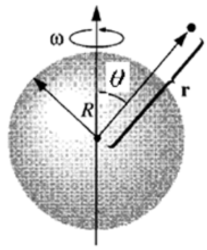
For surface current distributions, the magnetic vector potential can be written as $\mathbf{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{K}}{\imath}\,da'$, where **K** is the surface current



(a) Show that the magnetic vector potential **inside** a sphere of radius R, carrying a uniform surface charge $\boldsymbol{\sigma}$, spinning at angular velocity $\boldsymbol{\omega}$, is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}) = \frac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}},$$

(b)Show that the magnetic field inside the sphere is



$$\mathbf{B} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}) = \frac{2}{3}\mu_0 R\sigma\omega\ \hat{z}$$

Hint:
$$\int_{-1}^{+1}\frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$
$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\Big].$$
$$= 2r/3R^2 \ \text{ If } r \le R$$

$$\nabla\times\bar{A} = \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\Big]\hat{r} + \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\Big]\hat{\theta} + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\Big]\hat{\phi}$$

Part III: Electromagnetic Waves (50 points)

(1) Use the Lorentz force law and Maxwell equations to work out the dimension of the following quantities in terms of charge q, mass m, length ℓ and time t:
(a) electric displacement $\vec{D}$, (b) current density $\vec{J}$, (c) magnetic field $\vec{H}$,
(d) electric field $\vec{E}$, (e) magnetic induction $\vec{B}$, (f) poynting vector $\vec{S}$,
(g) electromagnetic momentum $\vec{D} \times \vec{B}$, (h) Maxwell tress tensor M_{ij}.

(2) Use the Lorentz force law and four Maxwell equations to point out
(a) how the quantities in (1) change under parity transformation $\vec{x} \to \vec{x}' = -\vec{x}$;
(b)and how they change under time reversal $t \to t' = -t$.

(3) Use spatial rotation, parity and time reversal (a) to pick up allowed terms of the following model for polarization (c_1, c_2 are scalars; α^{ij}, β^{ijk}, γ^{ijk} etc. are tensors):
$\frac{1}{\epsilon_0} P^i = \alpha^{ij} E^j + \beta^{ijk} E^j E^k + \gamma^{ijk} E^j B^k + \delta^{ijk} B^j B^k + c_1 (\vec{E} \times \dot{\vec{B}})^i + c_2 (\ddot{\vec{E}} \times \vec{B})^i$.
(b) For the allowed terms, what are generic expressions for these tensors?

(4) (a) Obtain the 2nd order differential equation for magnetic induction $\vec{B}(t, \vec{x})$ in medium with permeability μ and permittivity ϵ from $\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$ and $\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0$ (assume medium is linear, isotropic and homogeneous).
(b) Show the sources, $\rho(t, \vec{x}) = -\theta(t)\, \vec{p} \cdot \vec{\nabla} \delta^3(\vec{x})$ and $\vec{J}(t, \vec{x}) = \vec{p}\, \delta(t) \delta^3(\vec{x})$, satisfy current conservation. (c) Compute the magnetic induction with the sources identified in (b) using retarded Green's function $\frac{\delta(t - t' - \frac{1}{c}||\vec{x} - \vec{x}'||)}{||\vec{x} - \vec{x}'||}$.

EM

Part III (50 points)

(1) Prove the Poynting's theorem (15%)

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}. \tag{1}$$

U is the electromagnetic field energy density, and $\mathbf{S}$ is the Poynting vector.

(2) Prove that the propagation velocity of electromagnetic wave in any materials is slower than the light velocity in vacuum (15%).

(3) Consider a square waveguide of length a enclosed by a perfect conductor, in which it is composed of the non-dissipative materials with dielectric constant ϵ and magnetic permeability μ. (a) Derive the Helmholtz equations for the electric and magnetic fields from the Maxwell equations (10%), and (b) calculate the axial magnetic field of the threshold TE wave (10%).

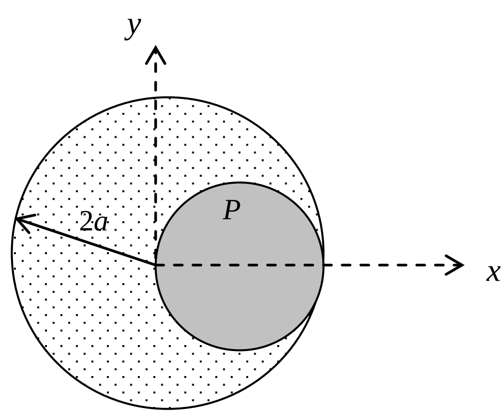
Electrodynamics, Part I — Qualifying Exam, March 24, 2017

1. **(10 Points)** Assuming that the electric field intensity $\boldsymbol{E}(x,y,z) = 10x\boldsymbol{i} + 10y\boldsymbol{j} + 10\,\boldsymbol{k}$ (V/m), find the total electric charge contained inside a cubical volume of 20 cm on a side centered symmetrically at the origin.

2. **(15 Points)** A point charge q is embedded at the center of a sphere of linear dielectric material (electric permittivity ε and radius a). (a) Find the electric displacement $\boldsymbol{D}$, the electric field $\boldsymbol{E}$, and the polarization $\boldsymbol{P}$ inside the sphere. (b) What is the polarization surface charge density?

3. **(10 Points)** An electric dipole $\boldsymbol{p}$ with a magnitude of 2×10^{-27} C-m is initially located at an origin along the z-axis. (a) Determine the <u>rotational torque</u> on this dipole when an electric field $\boldsymbol{E} = 5\boldsymbol{x} + 10\boldsymbol{y}$ (V/m) is applied. (b) How much energy changes if $\boldsymbol{p}$ finally points to the direction of $\boldsymbol{E}$?

4. **(15 Points)** A sphere with the dotted region of radius $2a$ has a uniform charge density ρ_o while the uniform charge density inside the small spherical part of radius a is $2\rho_o$. Find the <u>magnitude and direction of $\boldsymbol{E}$ field</u> at the position $P(a, a/2, 0)$.

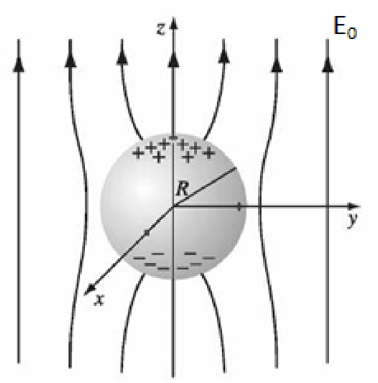


Part Ⅱ (50 points)

(1) A dielectric sphere of radius R with dielectric constant $\varepsilon/\varepsilon_0$ placed in an initially uniform electric field, which at large distances from the sphere is directed along the z axis and has magnitude of E_0, as shown below. Both inside and outside there are no free charges.

(a) Find the electric potential and electric field everywhere. (15 points)

(b) Calculate the induced polarization and surface charge density of the dielectric sphere. (10 points)



(2) A long cylindrical conductor (of radius a) has an off-center cylindrical hole (of radius a/2) down its full length, as shown in the following figure. If a current I flows through the conductor into the page, show that

(a) The volume current density flow through the conductor is

$$J = \frac{4I}{3\pi a^2}$$ (5 points)

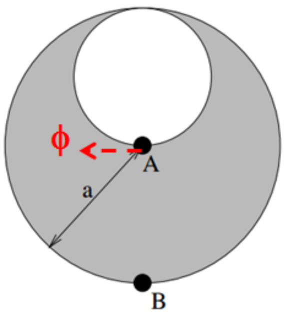
(b) the strength and direction of the magnetic field in point A is

$$\mathrm{B} = \frac{\mu_0 I}{3\pi a}(-\hat{\phi})$$ (10 points)

(c) the strength and direction of the magnetic field in point B is

$$\mathrm{B} = \frac{5\mu_0 I}{9\pi a}\hat{\phi}.$$ (10 points)

(hint: using superposition principle and Ampere's law)



Part Ⅲ (50 points)

(1) By using the Maxwell equations, the wave equations for the fields with given charge (ρ) and current (J) densities are described by (10 points)

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{-1}{\epsilon_0}\left(-\boldsymbol{\nabla}\rho - \frac{1}{C^2}\frac{\partial\mathbf{J}}{\partial t}\right)$$

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu_0\boldsymbol{\nabla}\times\mathbf{J}$$

The electromagnetic fields for the above equations are (10 points)

$$\mathrm{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\mathrm{R}}{\mathrm{R}^3}[\rho(\mathbf{x}',t')]_{\mathrm{ret}} + \frac{\mathbf{R}}{c\mathrm{R}^2}\left[\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\right]_{\mathrm{ret}} - \frac{1}{c\mathrm{R}^2}\left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\mathrm{ret}}\right\}$$

and

$$\mathrm{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\left\{[\mathrm{J}(\mathbf{x}',t')]_{\mathrm{ret}}\times\frac{\mathbf{R}}{\mathrm{R}^3} + \left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\mathrm{ret}}\times\frac{\mathbf{R}}{c\mathrm{R}^2}\right\}$$

where $\mathbf{R}=\mathbf{x}-\mathbf{x}'$ and $\mathrm{R}=|\mathbf{x}-\mathbf{x}'|$.

(2) The frequency-dependent dielectric constant $\epsilon(\omega)/\epsilon_0$ is an analytic function of ω in the upper half − plane. Show that its real and imaginary parts satisfy the Kramers-Kronig relations (15 points).

(3) We consider the propagation of TM waves in a rectangular waveguide with length a and width b, as shown in Fig. 1. The metallic waveguide is filled with dielectric constant ϵ and permeability μ. Calculate the electromagnetic fields, the cutoff frequency, and the field energy per unit length (15 points).

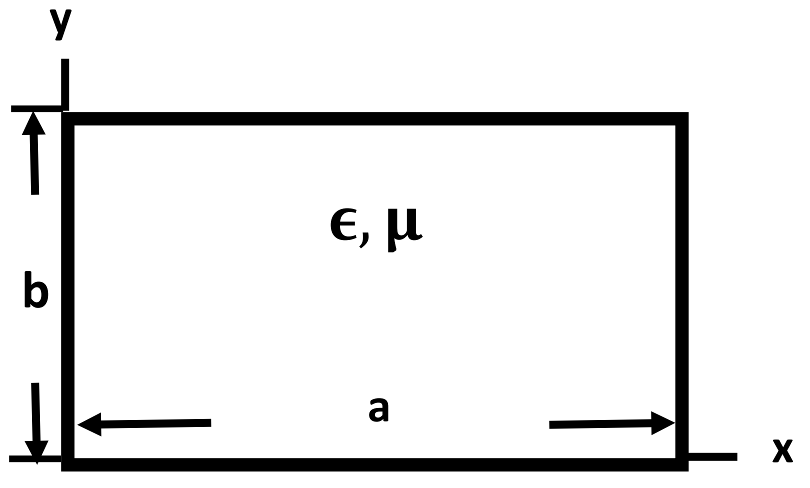


Fig. 1

Electrodynamics Part I (50 points)

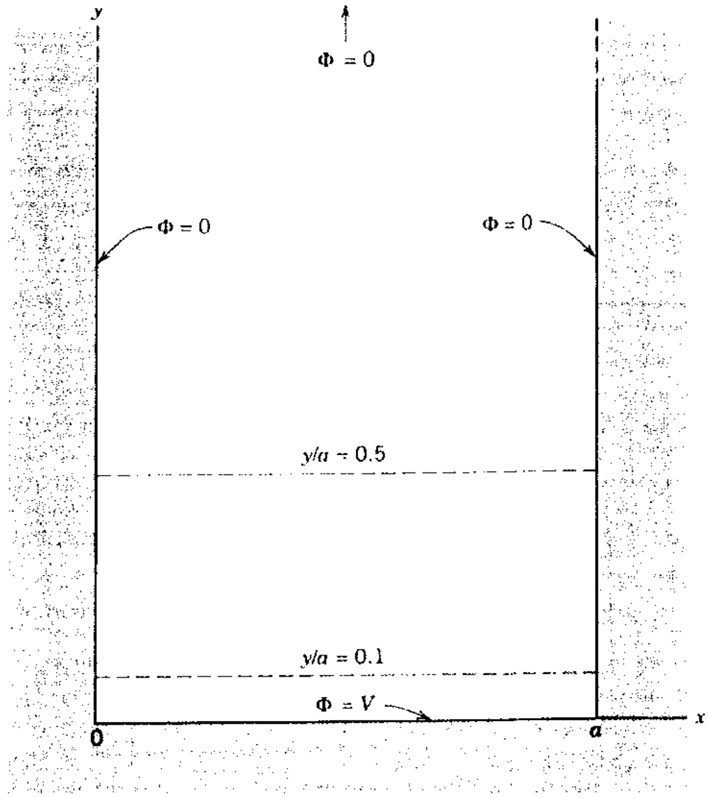


FIG 2

1. Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.
(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R. [7 pts]
(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b. [7 pts]
(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R. [6 pts]

2. In Figure 2, find the potential in the region, $0 \le x \le a$, $y \ge 0$, under the boundary conditions that $\Phi = 0$ at $x = 0$ and $x = a$, while $\Phi = V$ at $y=0$ for $0 \le x \le a$ and $\Phi \to 0$ for large y. [10 pts]

3. Three point charges (q, $-2q$, q) are located in a straight line with separation a and with the middle charge $-2q$ at the origin of a grounded conducting spherical shell of radius b, as indicated in Fig. 3.

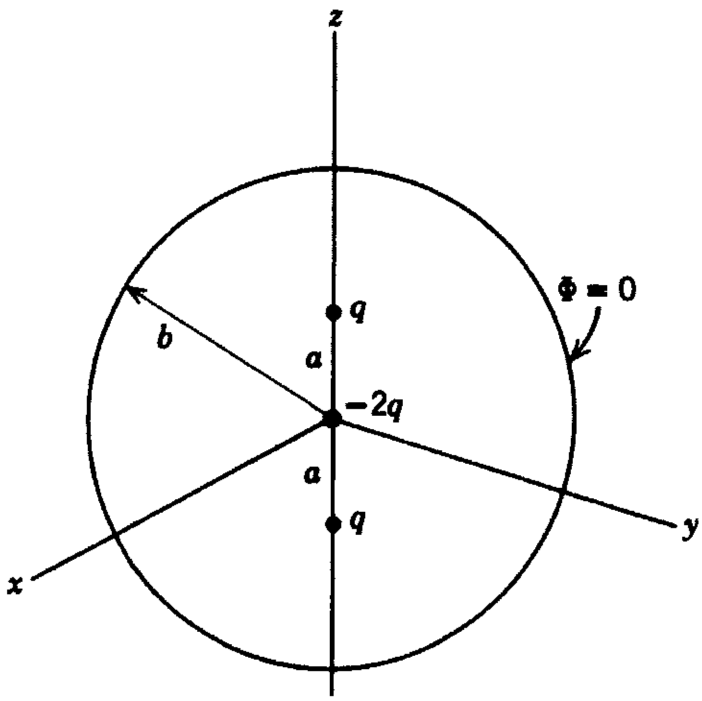


FIG 3

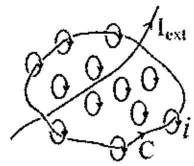
(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates. [10 pts]

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\varphi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta) \quad [10\text{pts}]$$

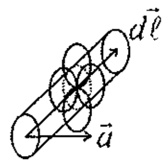
Electrodynamics - Part 2 (50 points)

Problem 1: (15 points)



In medium, we can represent each molecule by a current loop and its dipole moment by $\vec{m} = i\vec{a}$. Assume the number of molecules (loops) per unit volume is N, then the magnetic dipole density is $\vec{M} = Ni\vec{a}$.

By applying the Ampere's law in vaccum $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 \int_S \vec{J}\cdot d\vec{A} = \mu_0 I$



(S is any surface bounded by C), show that in medium

$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0\left(I_{ext} + \oint_C \vec{M}\cdot d\vec{\ell}\right)$, where I_{ext} is the external current. (Note that this is the integral form of $\vec{\nabla}\times\left(\vec{B}/\mu_0 - \vec{M}\right) = \vec{J}_{ext}$.)

Problem 2: (15 points)

The mutual inductances between two current loops are equal, i.e. $M_{12} = M_{21} \equiv M$. Now we have a very small and a large current loops which are originally far apart, each maintained at fixed currents i & I respectively by batteries. The small loop, with magnetic dipole $\vec{m} = i\vec{a}$, is then brought to the vicinity of the large loop, where the magnetic field is $\vec{B}(\vec{r})$. By carefully considering the energy supplied by the batteries to maintain the currents and the work done by the torque $\vec{m}\times\vec{B}(\vec{r})$, show that the total change in the magnetic energy is $+\vec{m}\cdot\vec{B}$ (not $-\vec{m}\cdot\vec{B}$).

Problem 3: (10 points)

It is well known that the magnetic field by an infinitely long and dense solenoid carrying a current is uniform $\vec{B} = B\hat{z}$ inside, and zero outside. Assuming the solenoid with a radius R, is circular, find the vector potential $\vec{A}$ everywhere (inside and outside) in terms of B & r, where r is the distance from the central axis of solenoid. In what gauge is your $\vec{A}$?

Problem 4: (10 points)

Derive the conservation of charge from the Maxwell's eqs.

成功大學物理研究所博士班 105 學年度上學期 [資格考試]

Electrodynamics (Part III, 50 points) 2016-10-21

1. Assuming that a monochromatic point source of N (Watt) is sending out a linearly polarized spherical wave into every direction.

   (a) Please write down the wave function of the spherical wave and show that your solution satisfies the wave equation. **(15 points)**

   (b) What is the momentum density of this spherical wave? **(8 points)**

   (c) Calculate the root mean square electric field at 1 meter away from the point source. **(7 points)**

2. A monochromatic electromagnetic plane wave is propagating toward the z-direction in a conductor, where the polarization of the electromagnetic plane wave is along x-direction. The wave number of the plane wave in a conductor is a complex number, i.e., $\tilde{k} = k + i\kappa$.

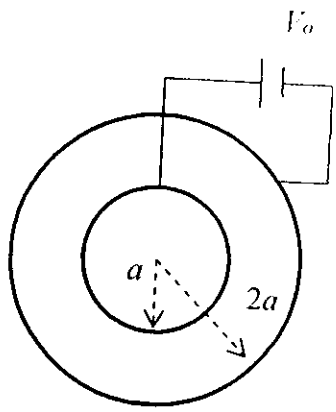
   (a) Determine the amplitude and the phase relation between the E-field and the B-field of the plane wave. **(10 points)**

   (b) Express B-field of the plane wave by the E-field amplitude E_0 and E-field phase δ_E. **(10 points)**

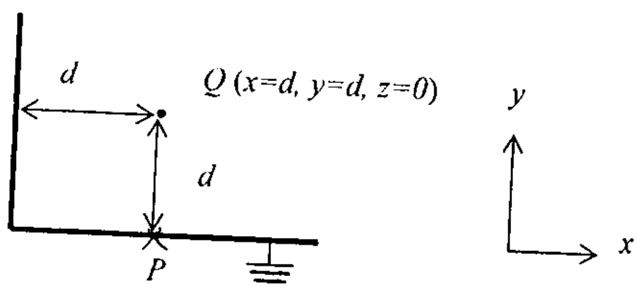
Electrodynamics, Qualifying Exam Part I

2016-03-25

1. **(25 Points)** Assuming that the space between the inner and outer conductors of a concentric sphere is filled with a uniform charge cloud having a volume charge density $\rho(r) = K/r$ for $a < r < b = 2a$, where a and b are the radii of the inner and outer conductors respectively. The inner conductor is maintained at a constant potential V_o, while the outer one is grounded. Determine the electric potential distribution in the region $a < r < b$ by solving the Poisson's equation.



2. **(25 Points)** A positive point charge Q is located at a distance d from two grounded perpendicular conducting half-planes, as shown below.
(a) Find the electric field intensity $\boldsymbol{E}$ (magnitude and direction) at the point $P(x = d, y = 0, z = 0)$.
(b) Determine the force (magnitude and direction) on Q caused by the charges induced on the planes.



**Useful formula:*

In spherical coordinates:

$$\nabla^2(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Classical Electrodynamics: II (50%)

(1) There are two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ at $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. Determine the arrangement with the lowest interaction energy and its energy. (10%)

(2) The electric dipole moment $\mathbf{p}$ is situated at the center of a sphere with radius R. Calculate the volume integral of the electric field over the sphere. (15%) Also noticed that there are useful mathematical equations in appendix.

(3) (a) Calculate the magnetic scalar potential Φ_M for a sphere of radius a, with a uniform permanent magnetization $\mathbf{M}$. (10%) (b) Now, the sphere is a linear magnetic substance, and its magnetic permeability is μ. Moreover, it exists in a uniform magnetic induction $\mathbf{B}_0$. Find the relation between $\mathbf{M}$ and $\mathbf{B}_0$. (5%)

(4) A very long cylinder of radius a carriers a current I. Calculate the magnetic inductance per unit length inside the cylinder. (10%)

Appendix: useful mathematical equations

$$\int_{r\leq R} \nabla\Phi(\mathbf{x})d^3x = \int_{r=R} \Phi(\mathbf{x})R^2\mathbf{n}d\Omega. \qquad (1)$$

$$\int \cos\gamma \; \mathbf{n}d\Omega = \frac{4\pi\mathbf{n}'}{3}. \qquad (2)$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma). \qquad (3)$$

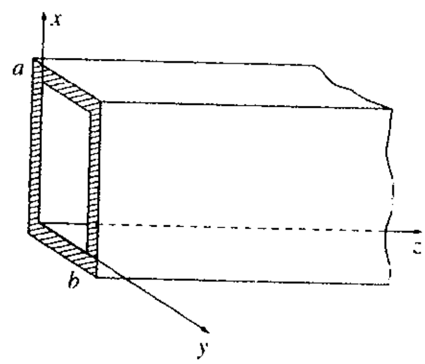
成功大學物理研究所博士班 104 學年度下學期 [資格考試]

Electrodynamics (Part III, 50 points) 2016-03-25

1. A monochromatic electromagnetic plane wave is propagating toward the z-direction in vacuum, where the polarization of the electromagnetic plane wave is along x-direction and the wavelength of the plane wave is λ.
   (a) Write down the expression of the E-field of the plane wave. **(3 points)**
   (b) Express B-field of the plane wave by the E-field amplitude and E-field phase. **(3 points)i**
   (c) Find the ***long-time-averaged value*** of three quantities of the plane wave: the energy density, Poynting vector, and momentum density. **(9 points)**

2. Write down the integral form of two important conservation laws of electromagnetism, i.e. the conservation law of electromagnetic energy and the conservation law of electromagnetic momentum, respectively. Clearly address the physical meanings of each term in two expressions. **(15 points)**

3. Suppose we have a wave guide of rectangular shape, with height a and width b (a>b). It is a hollow pipe and the wave guide is composed of perfect conductor. In the metal pipe, the longitudinal components E_z and B_z of the guided waves should fulfill the equations,



$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]E_z=0,$$
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]B_z=0.$$

Where the other components, E_x, E_y, B_x, and B_y, could be determined by the longitudinal components E_z and B_z of the guided waves, i.e.,

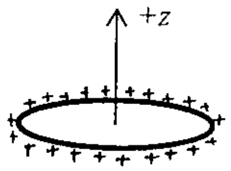
$$\text{(i)}\ E_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right),$$
$$\text{(ii)}\ E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right),$$
$$\text{(iii)}\ B_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right),$$
$$\text{(iv)}\ B_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).$$

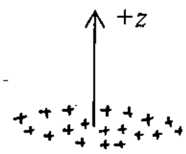
We are interest in the TM waves in the rectangular wave guide.
(a) Find the TM waves in the rectangular wave guide. **(8 points)**
(b) Find the lowest cutoff frequency of TM modes. **(4 points)**
(c) Find the phase and group velocity of the TM waves. **(8 points)**

成功大學物理研究所博士班 104 學年度[資格考試] 電動力學試題

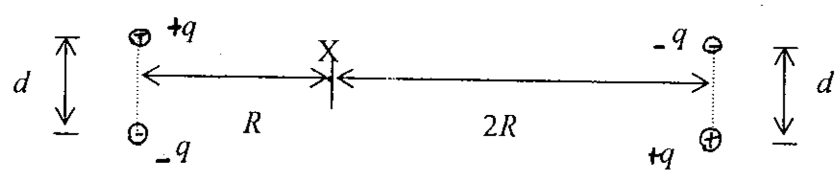
Electrodynamics (Part I, 50 points) 2015-10-16

1. **(15 Points)** (a) Calculate the electric potential $V(z)$ at a point z along the axis of a charged ring of radius a which contains a total charge Q. (b) Find the z-component of the electric field (E_z) at the point z. (c) Where is the maximum E_z for this charged ring? (d) From the result of (a), calculate the electric potential $V(z)$ at a point z along the axis of a charged disk of radius a which contains a total charge Q. (e) Find the z-component of the electric field (E_z) at the point z. (f) Where is the maximum E_z for this charged disk?

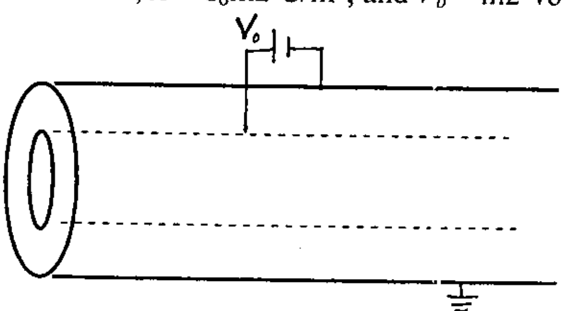




2. **(20 Points)** (a) Evaluate the electrostatic energy for the configuration of four charged particles shown below. (b) Compute the electric potential at the position X. (c) Calculate the electric field E (magnitude and direction) at point X. (d) If a charged particle $-Q$ is placed at the position X, what is the electric force experienced by that particle? You should provide the magnitude and direction for the force in your answer. (e) Compute the electric field intensity $\boldsymbol{E}$ at position X as $R >> d$. Express your answer in terms of the definition of electric dipole $\boldsymbol{p}$.



3. **(15 Points)** Assuming that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density charge $\rho(r) = A/r$ for $a < r < 2a$, where a and $2a$ are the radii of the inner and outer conductors respectively. The inner conductor is maintained at a constant potential V_o while the outer conductor is grounded. Determine the potential distribution in the region $a < r < 2a$ by solving the Poisson's equation with numerical values of $a = 2$ m, $A = \varepsilon_0 \ln 2$ C/m^2, and $V_o = \ln 2$ Volt. (Here $\varepsilon_0 = 1/(36\pi \times 10^9)$ F/m)



In cylindrical coordinates,

$$\nabla^2(r,\phi,z) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2}$$

Electrodynamics (Part II 50 points).

1. A solenoid consists of n turns per unit length wrapped around a cylindrical tube of radius a and carries current I (Fig. 1).
   (a) As shown in Fig. 1, find the magnetic field at point P on the axis of the solenoid. Express your answer in terms of θ_1 and θ_2. (10 pts)
   (b) If the solenoid is infinite long, find the field on the axis of the infinite solenoid. (5 pts)
   (c) Find the vector potential $\vec{A}$ inside and outside an infinite solenoid. (10 pts)

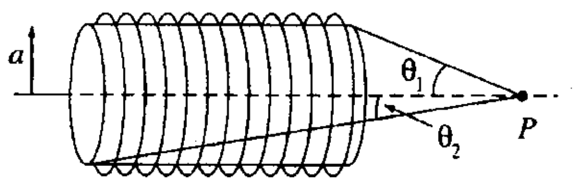


Fig. 1

2. (a) Is $\vec{\nabla}\cdot\vec{H}$ always equal to zero in magnetostatic cases? If not, provide an example to show $\vec{\nabla}\cdot\vec{H}\neq 0$. (5 pts)
   (b) Can you always express $\vec{H}$ as $-\vec{\nabla}\cdot V_M$? where V_M is a magnetic scalar potential. If not, please describe the required conditions. (5 pts)
   (c) As shown in Fig. 2, a very large piece of hard magnetic material has magnetization $\vec{M}$. The B-field inside the material is $\vec{B}_0$, and the H-field inside the material is $\vec{H}_0$. Now a cavity is hollowed out of the material. Find the B- and H- fields at the center of the cavity for three cases: (10 pts)
   (i) The cavity is a small sphere.
   (ii) The cavity is a long needle running parallel to $\vec{M}$.
   (iii) The cavity is a thin wafer perpendicular to $\vec{M}$.

   Express the answer of B-fields in terms of $\vec{B}_0$ and $\vec{M}$, and H-fields in terms of $\vec{H}_0$ and $\vec{M}$.

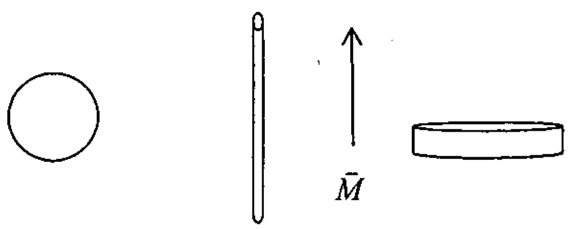


Fig. 2

3. Can a static magnetic field exist inside a perfect conductor? Explain why or why not. (5 pts)

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Electrodynamics (Part III, 50 points) 2015-10-16

1. The Maxwell equations can be rephrased as two equations,

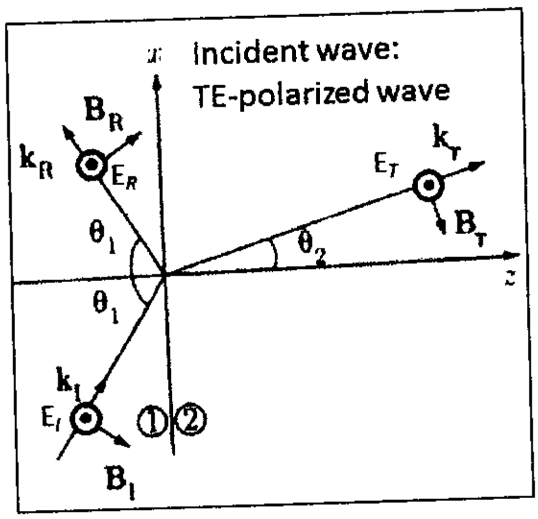
$$\begin{cases} \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\dfrac{1}{\epsilon_0}\rho \\ \left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \dfrac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \end{cases}, \quad \text{.........(※)}$$

which contains all the information in Maxwell's equations.

(a) What is the gauge transformation? **(10 points)**

(b) Show that in the Lorentz gauge the addressed two equations (※) can be rephrased in the form of two independent inhomogeneous wave equations. **(10 points)**

2. Referring to the figure, an electromagnetic wave of polarization perpendicular to the plane of incidence is obliquely impinging on the boundary. Medium 1 and 2 are both dielectrics. The incident, reflected, transmitted waves are represented as following:



$$\begin{cases} \tilde{\mathbf{E}}_I = \tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I = \dfrac{1}{v_1}\tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}(-\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{cases}$$

$$\begin{cases} \tilde{\mathbf{E}}_R = \tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R = \dfrac{1}{v_1}\tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}(\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{cases}$$

$$\begin{cases} \tilde{\mathbf{E}}_T = \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T = \dfrac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_2\,\hat{\mathbf{x}} + \sin\theta_2\,\hat{\mathbf{z}}); \end{cases}$$

(a) Write down boundary conditions of E-field and B-field. **(10 points)**

(b) Impose the boundary conditions, and obtain the Fresnel equations for E_{0R} and E_{0T}. **(20 points)**