Part I (50 points)

1. Consider a quantum mechanical system which contains two complete and orthonormal energy eigenstates, $|1\rangle$ and $|2\rangle$. Three observables, P, Q, and R are measured in the system. $|1\rangle$ and $|2\rangle$ may or may not be the eigenstates of P, Q, and R. Based on the following experimental data, (a) find all the eigenvalues of P, Q, and R as possible as you can. (10 points) (b) One group of the experimental data is unreasonable. Which group is it? (5 points)

   (i) $\langle 1|P|1\rangle=\frac{1}{2}$, $\langle 1|P^2|1\rangle=\frac{1}{4}$;

   (ii) $\langle 1|Q|1\rangle=\frac{1}{2}$, $\langle 1|Q^2|1\rangle=\frac{1}{6}$;

   (iii) $\langle 1|R|1\rangle=1$, $\langle 1|R^2|1\rangle=\frac{5}{4}$, $\langle 1|R^3|1\rangle=\frac{7}{4}$.

2. The Hamiltonian operator of a particle in one dimension is given by $H=\frac{p^2}{2m}+V(x)$.

   From the commutation relation $[x, p]=i\hbar$, prove that
   $$\sum_n (E_n-E_0)|\langle n|x|0\rangle|^2 = constant$$
   where E_n and E_0 are the energy eigenvalues of eigenstates $|n\rangle$ and $|0\rangle$, respectively. Also find the value of the constant. (15 points)

3. Consider a one dimension oscillator whose Hamiltonian is given by $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2 x^2$.

   (a) Find the dependence of the expectation value on time for the initial position and initial momentum operators:

   $x_0=x\cos\omega t-\frac{p}{m\omega}\sin\omega t$;

   $p_0=p\cos\omega t-m\omega x\sin\omega t$. (5 points)

   (b) Does the x_0 or p_0 operator commute with the Hamiltonian operator? Also discuss if the results conflict with results in (a). (10 points)

   (c) In Heisenberg picture, what are the equations of motion for operators x_0 and p_0? (5 points)

1. Consider a system of four distinguishable particles, each with spin ½. In this problem all degrees of freedom other than the spins are to be ignored, and $\hbar=1$. Let $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$, and $\hat{s}_4$ be the operators for spins of the particles.

   (a) What is the eigenvalues of $\hat{s}_1^2$? [10 points]

   (b) How many states are needed to make a complete orthogonal basis set for this system? [10 points]

   (c) What are the possible eigenvalues of $\hat{S}^2$, where
   $$\hat{S}=\hat{s}_1+\hat{s}_2+\hat{s}_3+\hat{s}_4$$
   is the operator for the total spin. [10 points]

2. Consider a system consisting of a spin-1/2 particle with a gyromagnetic ratio, γ, placed in a uniform magnetic field. The magnetic field is given as $\vec{B}$ with the component $(B_x,0,0)$, where B_x is a constant and does not vary with time. All degrees of freedom other than the spins are to be ignored. Note that the gyromagnetic ratio, γ, is defined as
   $$\vec{\mu}=\gamma\vec{S},$$
   where $\vec{\mu}$ and $\vec{S}$ are the magnetic moment and the spin operator, respectively. Not that $\vec{S}$ for a spin-1/2 particle can be expressed in terms of the Pauli operators:
   $$\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \sigma_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

   (a) Write down the Hamiltonian that describes the interaction of this spin-1/2 particle with the magnetic field.[2 points]

   (b) At $t = 0$, the spin-1/2 particle is prepared with its spin pointing along the positive z direction. What is the probability of finding the particle's spin pointing along the negative z direction for $t > 0$? [10 points]

   (c) Show by giving a derivation that the time evolution of the spin is described by the equation
   $$\frac{d\vec{S}}{dt}=\gamma\vec{S}\times\vec{B}. \quad \text{[8 points]}$$
   [This equation is particularly easy to derive in the Heisenberg representation, but you may derive it in any representation you like.]

Part III: 50 %

1. **[5%] The scattering amplitude of a scattering process has the form**

$$f(\theta, \phi) = -b_1 + ib_2\,,\ \ b_1 \text{ and } b_2 \text{ are real constants.}$$

   What is the total cross-section σ_{total}?

2. **[15%] Obtain the differential scattering cross section in the first Born approximation for the spherically-symmetric potential $V = -V_0 e^{-\alpha r}$, in terms of the momentum transfer q and the scattering angle θ. [You may find this integral useful: $\int_0^\infty dx x^n e^{-Ax} = n!/A^{n+1}$.]**

3. **[10%] A scattering potential has translational invariance property $V(\vec{r}) = V(\vec{r} + \vec{R})$, where $\vec{R}$ is a constant vector. Find the condition for non-vanishing first Born scattering amplitude.**

4. **[20%] Consider a system with a simple 2×2 matrix Hamiltonian $H = H_0 + H_1$, for which**

$$H_0 = \begin{vmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{vmatrix} \quad \text{and} \quad H_1 = \begin{vmatrix} 0 & v \\ v^* & 0 \end{vmatrix}\,.$$

   First find the exact energies of the system $(H_0 + H_1)|\Psi\rangle = E|\Psi\rangle$, then use perturbation theory to find the energies for the nondegenerate and degenerate cases: $\epsilon_1 \neq \epsilon_2$ and $\epsilon_1 = \epsilon_2$.